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CONTENTS

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THE UBIQUITOUS 3:4:5 TRIANGLE

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and

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While studying various relationships between arcs and circles connected with the square, we were intrigued by the frequent unexpected appearance of 3: 4: 5 triangles, some of which are shown in the squares of Figure 1. The ancient Pythagoreans would have been delighted to have this evidence to support their number mysticisms.

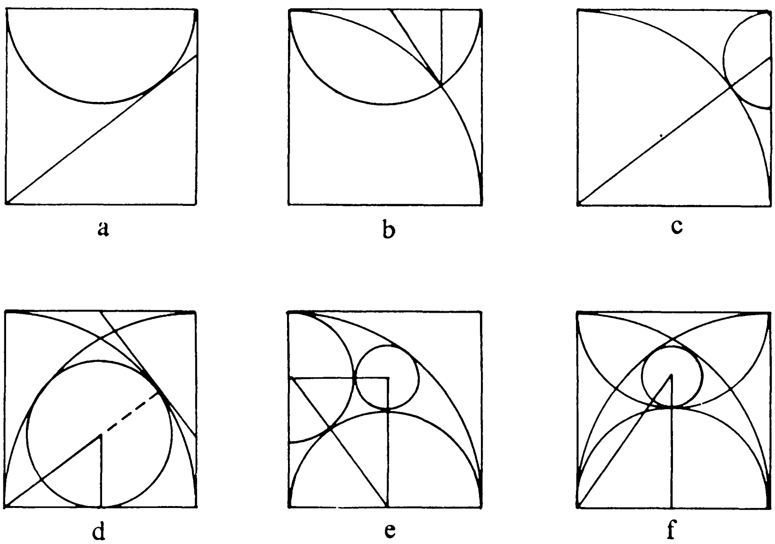

FIG. 1

Not being mystics, we have sought and found some underlying patterns which generate 3: 4: 5 triangles. However, many of these triangles occur in isolated situations.

A linear network. Figure 1a is the starting pattern of Figure 2 — a square $ABCD$ with semicircle (F) on DC and its tangent AG extended to meet BC at H. (Throughout this discussion, circles and circular arcs are identified in the conventional fashion by the letters at their centers in parentheses.) For convenience, the side of the square is taken to be 24, so $DF = FC = FG = 12$.

The line joining the midpoints F and E of DC and AB, respectively, intersects the semicircle (F) in Q and AH in K. The radius FG extended meets CB in J and is perpendicular to AH. The tangents $HG = HC$ and $AG = AD = 24$. Right triangles FHG and FHC are congruent as are FAG and FAD, so the hypotenuses bisect angles GFC and GFD. Hence, AFH is a right triangle and $HG = (FG)^2/AG = (12)^2/24 = 6 = HC$.

Then $HB = 18$, $AH = 30$, $AK = 15$, $KG = 9 = KE$, $FK = 15$, and $QK = 3$. Since triangles FGK and JGH are similar, $HJ = 10$, $JB = 8$, $GJ = 8$, $FJ = 20$, and $CJ = 16$. DJ intersects FE at I, so $FI = 8$ and $IQ = 4$.

It follows that triangles FCJ, HGJ, FGK, AKE, and ABH [Figure 1a] are 3: 4: 5 triangles.

Note that the side BC of the square has been divided in the ratio 4: 5: 3.

A parallel to BC through G cuts FC in P. Triangles FCJ and FPG are similar, so the latter also is a 3: 4: 5 triangle.

A parallel to AB through I meets DA in M and produces a 12-by-16 rectangle $IMAE$. Consequently EMI and AIE are 3: 4: 5 triangles, with $ME = 20 = AI$.

Clearly, the network may be extended to develop any desired number of 3:4:5 triangles by drawing parallels or perpendiculars to the sides of those already identified, for example, the arrow-headed lines in Figure 2. Or, the configuration may be reflected about EF to generate more such triangles.

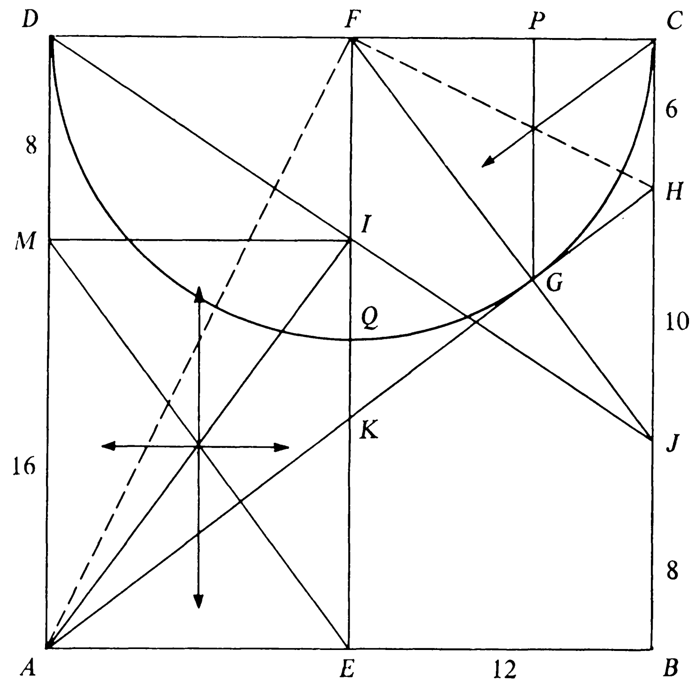

FIG. 2

The related circular network. In the circular network of Figure 3, the essential points are lettered in the same manner as in Figure 2. Present are quadrants (A), (B), (D), radius 24; semicircles (E), (F), radius 12; semicircle (M), radius 8; semicircle (H), radius 6; circle (K), radius 9; and circle (I), radius 4. Lengths and angles developed in the previous section are employed below.

$DA = AG = AB = 24$, so G is the intersection of (A) and (F), [Figure 1b], and FPG is a 3: 4: 5 triangle.

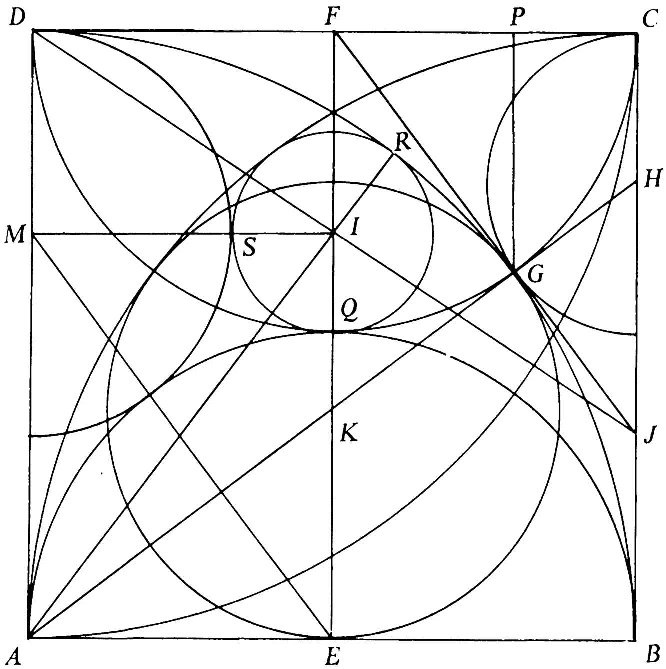

FIG. 3

$HG = HC = 6$, AH and FJ are perpendicular, so (A) and (H) are tangent, [Figure 1c], and AHB is a 3: 4: 5 triangle.

A, K, and G are collinear. $KE = KG$, so (K) is tangent to AB, to (A), and by symmetry to (B), [Figure 1d]. AKE and FCJ are 3: 4: 5 triangles.

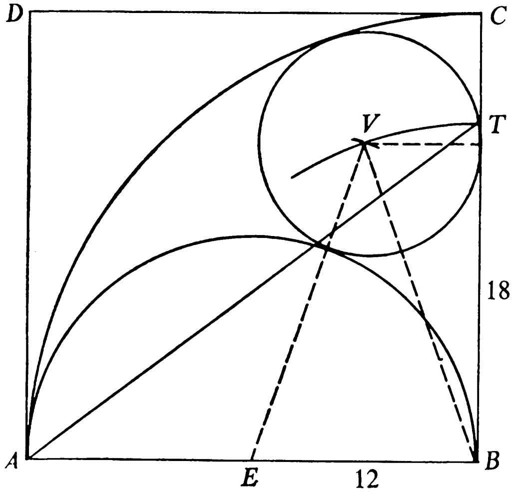

FIG. 4

$ME = 20 = 8 + 12$, so (M) and (E) are tangent. $IS = IM - SM = 12 - 8 = 4 = IQ$, and $IR = AR - AI = 24 - 20 = 4$, so (I) is tangent to (M), (A), (B), (E) and (F), [Figures 1e and 1f]. EMI and AIE are 3: 4: 5 triangles.

Three associated configurations. In Figure 4 it is desired to inscribe a circle (V) in the mixtilinear triangle bounded by (E), (B), and CB. If the radius of the inscribed circle be r, then $VE = 12 + r$ and $VB = 24 - r$. If these be equated, $r = 6$, which is the distance of V from CB. The center, V, of the required inscribed circle lies at the intersection of circles with radii 18 and centers at E and B. The second of these circles cuts BC at T, so ABT is a 3: 4: 5 triangle.

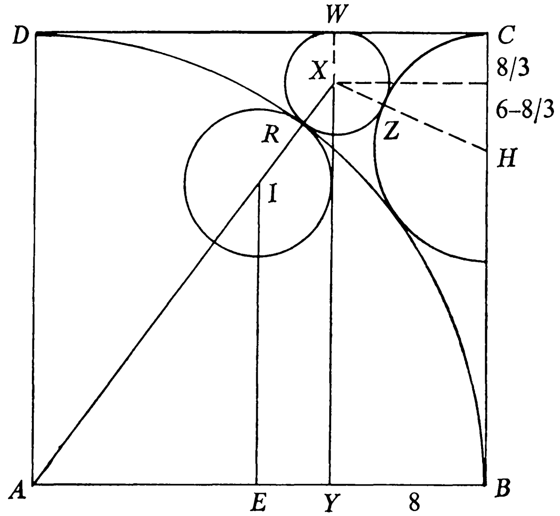

FIG. 5

From Figure 3, (A), (H) and (I) are isolated in Figure 5. When AI is extended to intersect the tangent to (I) which is parallel to AD, the 3: 4: 5 triangle AXY similar to AIE is obtained. Since $AY = 12 + 4 = 16$, then $XY = 64/3$ and $AX = 80/3$. Now $ZH = CH = 6$. Hence, $RX = 8/3 = WX$ and $XZ = XH - ZH = \sqrt{8^2 + (6 - 8/3)^2} - 6 = 8/3$. Consequently, X is the center of a circle tangent to (A), (H), and DC.

In Figure 6, the circle (G) with radius r is inscribed in the mixtilinear triangle bounded by AB, (E), and (F), which have radii of 12. Now $(FJ)^2 + (JG)^2 = (FG)^2$ and $(EH)^2 + (GH)^2 = (EG)^2$. Whereupon, placing $EH = x$,

$$(12 - r)^2 + (12 + x)^2 = (12 + r)^2 \text{ and } x^2 + r^2 = (12 - r)^2.$$

Eliminating r between these last two equations gives

$$x^2 + 8x - 48 = 0$$

which has the positive root $x = 4$. Therefore, $r = 16/3$.

It follows that EGH, FHA, FNJ, and HNG are 3: 4: 5 triangles, and FH is

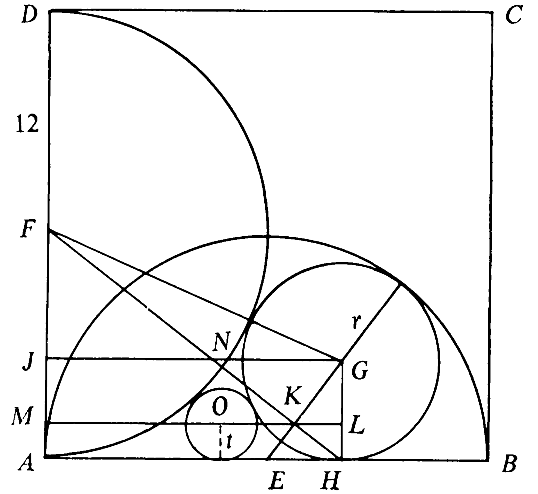

FIG. 6

perpendicular to EG at K. An extensive network of 3: 4: 5 triangles can be constructed by drawing perpendiculars from the vertices of the right angles to the hypotenuses of those already identified. Thus among the 3: 4: 5 triangles in Figure 6 are GHK, EHK, KGL, and KHL with $KH = 16/5$ and $LH = 48/25$.

Now the reciprocal of the square root of the radius of a circle tangent to two circles and to their common tangent is equal to the sum of the reciprocals of the square roots of the radii of these circles [**1**]. Thus if the radius of the circle inscribed in the mixtilinear triangle bounded by AH, (F), and (G) be t, then $1/\sqrt{t} = 1/\sqrt{12} + 1/\sqrt{16/3}$, whereupon $t = 48/25 = LH$.

Four more configurations. In Figure 7, the center of each of two equal circles (P) and (Q) lies on the circumference of the other. DE is the line of centers extended.

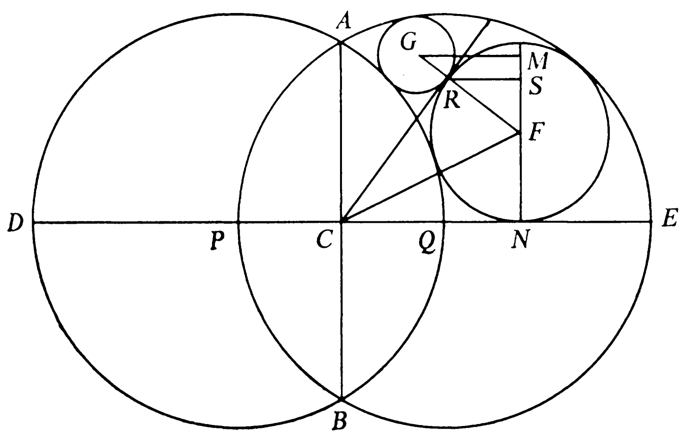

FIG. 7

AB is the common chord and radical axis of (P) and (Q); (F) is inscribed in the mixtilinear triangle bounded by arcs QA and AE and the segment EQ. The circle (G) is inscribed in the smaller curvilinear triangle bounded by (P), (Q), and (F).

If a variable circle touches two fixed circles, its radius has a constant ratio with the perpendicular from its center to their radical axis [**2**]. (G) and (F) are two positions of the variable circle. Another would have QE as a diameter, so the constant ratio is $1:2$, and $CN = 2FN$.

If two circles touch two others, and belong to the same series, the radical axis of either pair passes through the corresponding center of similitude of the other pair [**3**]. C is the internal center of similitude of (P) and (Q), so lies on the common internal tangent, CR, of (G) and (F); CR is perpendicular to GF; FN is perpendicular to CN and GM is perpendicular to NF extended. $CR = CN$, so $\angle MFG = \angle NCR = 2\angle NCF$. $\operatorname{Tan} NCF = \frac{1}{2}$, so $\tan MFG = \tan NCR = 4/3$, whence triangle FMG is a 3: 4: 5 triangle, as is FSR.

LEMMA. *In a right triangle with $a^2 + b^2 = c^2$, if $c - b = b - a$, then the sides are in the ratios* 3: 4: 5. Since $c = 2b - a$, then $a^2 + b^2 = 4b^2 - 4ab + a^2$, or $b(3b - 4a) = 0$. Hence, $a = 3k$, $b = 4k$, $c = 5k$.

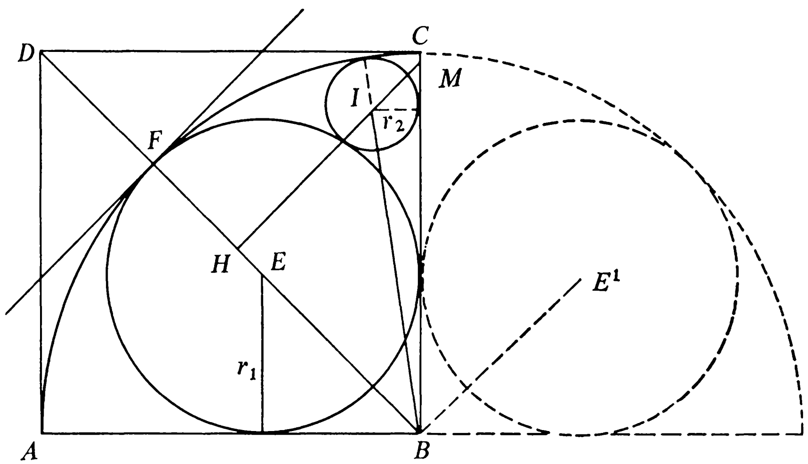

FIG. 8

In Figure 8, the quadrant (B) with radius R is inscribed in square $ABCD$. Circle (E), radius r_1, is inscribed in the quadrant; and circle (I), radius r_2, is tangent to (E), (B), and CB. The line of centers BE extended meets the circles at F. The perpendicular to BF at F is tangent to (B) and (E) and is their radical axis. HI, a perpendicular to FB, when extended meets CB in M; (B) and (E) are reflected about CB.

Considering (E') and (I) to be two positions of a variable circle, and applying the previously mentioned theorem of Casey [**2**], $r_1 : FB = r_2 : FH$. Now $R = FB = FE + EB = r_1(1 + \sqrt{2})$, so $FH = FB(r_2/r_1) = r_2(1 + \sqrt{2})$. Furthermore, $HB - HI = HM - HI = IM = r_2\sqrt{2}$. Also, $IB - HB = (R - r_2) - [R - r_2(1 + \sqrt{2})] = r_2\sqrt{2} = HB - HI$. Therefore, by the lemma, IBH is a 3: 4: 5 triangle.

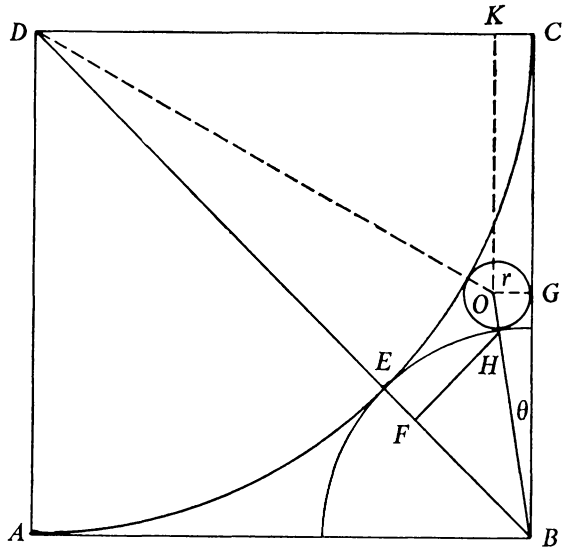

FIG. 9

Without loss of generality, the side of the square $ABCD$ in Figure 9 may be taken as 1, which is the radius of the quadrant (D). Then the radius of the quadrant (B) is $\sqrt{2} - 1$. (O) is tangent to BC, (B) and (D). Its radius $OG = OH = r$; OK is perpendicular to CD, and HF is perpendicular to DB. From triangle DKO, $CG = KO = \sqrt{(OD)^2 - (DK)^2} = \sqrt{(1+r)^2 - (1-r)^2} = 2\sqrt{r}$. From triangle GOB, $GB = \sqrt{(OB)^2 - (OG)^2} = \sqrt{(r + \sqrt{2} - 1)^2 - r^2}$. Also, $GB = CB - CG = 1 - 2\sqrt{r}$. Consequently, $r(3 - \sqrt{2}) - 2\sqrt{r} + \sqrt{2} - 1 = 0$, so $\sqrt{r} = (2\sqrt{2} - 1)/7$. It follows that $\tan\theta = OG/GB = 1/7$. Hence, $\tan HBF = \tan(\pi/4 - \theta) = 3/4$, so HFB is a 3: 4: 5 triangle [4].

On a line segment, AB, equal to $x^2 + y^2 = z^2$, a semicircle is constructed and a perpendicular DC is erected at the junction of the x^2 and y^2 segments. Then $CD = \sqrt{x^2y^2} = xy$, and $AC = xz$, $CB = yz$. Hence, the sides of right triangles ADC, CDB, and ACB in Figure 10 are in the ratios x: y: z.

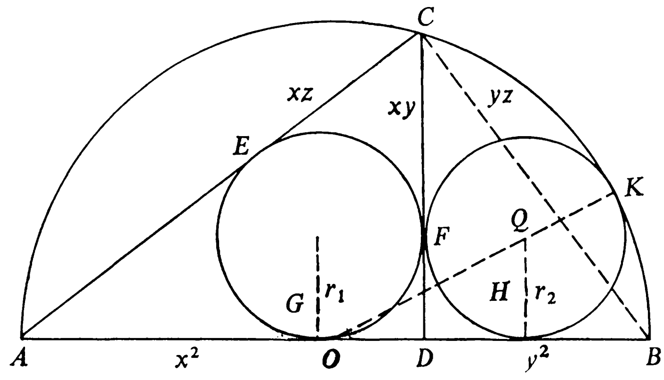

FIG. 10

Since $CE = CF$, $AE = AG$, and $DG = DF = r_1$, the inradius of triangle ACD is $r_1 = x(x+y-z)/2$.

The inradius r_2 of the mixtilinear triangle bounded by BD, DC, and the arc CB is a leg of the right triangle OHQ. Hence

$$r_2^2 + (AD + r_2 - AB/2)^2 = (OK - r_2)^2.$$

That is,

$$r_2^2 + (x^2 - z^2/2 + r_2)^2 = (z^2/2 - r_2)^2,$$

of which the positive root is $r_2 = x(z-x)$.

If $r_1 = r_2$, then $z - x = y/3$, and since $(z+x)(z-x) = y^2$, $3z = 5y$ and $4z = 5x$. Therefore $y : x : z = 3 : 4 : 5$.

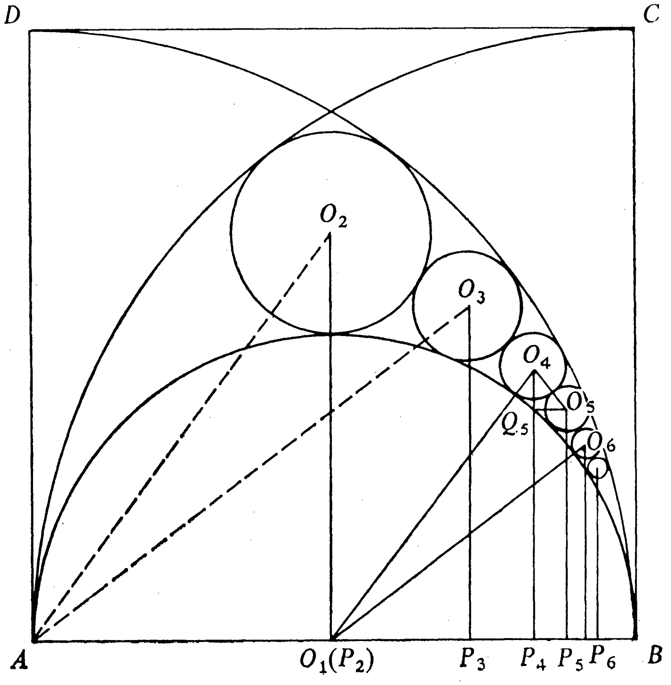

FIG. 11

A Steiner chain. In Figure 11, in the curvilinear triangle formed by the quadrants (A) and (B) with radius 2, and the semicircle (O_1) with radius 1, the maximum circle (O_2) is inscribed, and then a chain of Steiner circles (O_n), $n > 1$, is continued down into the horn angle with vertex B. The feet of the perpendiculars dropped from the centers O_n to BC are designated by P_n. Then O_1 and P_2 coincide. It has been shown [5] that the radius of (O_n) is $r_n = 2/(n^2+2)$. Also, that the sides of the right triangles $O_1O_nP_n$ are

$$O_1O_n = (n^2+4)/(n^2+2), \quad O_nP_n = 4n/(n^2+2), \quad \text{and} \quad O_1P_n = (n^2-4)/(n^2+2).$$

Consequently, $O_1O_4P_4$ and $O_1O_6P_6$ are 3 : 4 : 5 triangles.

It follows that $AP_n = 1 + O_1P_n = (2n^2-2)/(n^2+2)$ and $AO_n = (2n^2+2)/(n^2+2)$. Consequently, AP_2O_2 and AP_3O_3 are 3: 4: 5 triangles.

Now if Q_n is the foot of the perpendicular from O_n to $O_{n-1}P_{n-1}$, then

$$O_nQ_n = P_nO_1 - P_{n-1}O_1 = 6(2n-1)/(n^2+2)(n^2-2n+3),$$

$$O_{n-1}Q_n = O_{n-1}P_{n-1} - O_nP_n = 4(n+1)(n-2)/(n^2+2)(n^2-2n+3),$$

and the line of centers

$$O_{n-1}O_n = r_{n-1} + r_n = 2(2n^2-2n+5)/(n^2+2)(n^2-2n+3)\,.$$

Consequently $O_4O_5Q_5$ is a 3: 4: 5 triangle.

Network by folding. When the vertices of a square are connected to the midpoints of the nonadjacent sides, either by folding or by drawing, two overlapped four-pointed stars are formed, as in Figure 12. This pattern contains twenty-four 3: 4: 5 triangles, eight of each of the types BFA, GFE, and CDE [6].

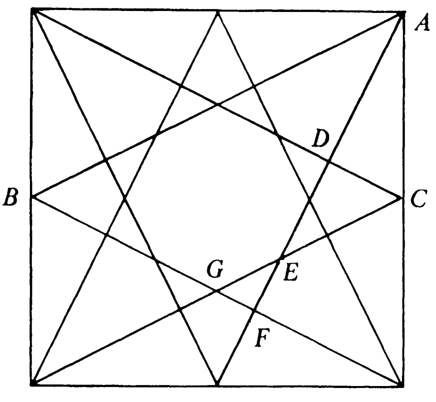

FIG. 12

Other appearances of 3: 4: 5. The 3: 4: 5 ratios show up again in the dissection of a triangle by straight cuts into four pieces which can be arranged to form two triangles similar to the given triangle. Take D, E, F on AB, BC, CA, the sides of any triangle ABC, such that $AD/AB = CE/CB = AF/AC = 1/5$. Let G be on AC with $AF = FG$, and H be the midpoint of DE. Then segments DE, DF, GH in Figure 13

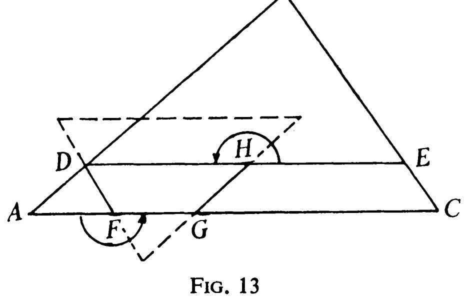

FIG. 13

represent the three cuts. One of the required triangles is BDE, the second triangle is formed by rotating ADF about F until A and G coincide, and then rotating the two-piece combination about H so that D and E coincide [7], [8]. Then the corresponding sides of the three triangles are in the ratios

$$(2DF + EC): BE: BC = 3:4:5.$$

When the side CD of square $ABCD$ is extended to E so that $CD = DE$, BE is drawn intersecting AD at F, and a line is drawn from C in the direction of A meeting BF at G as in Figure 14, then the square is dissected so that areas ABF: BGC: $GCDF$ = 3:4:5 [9].

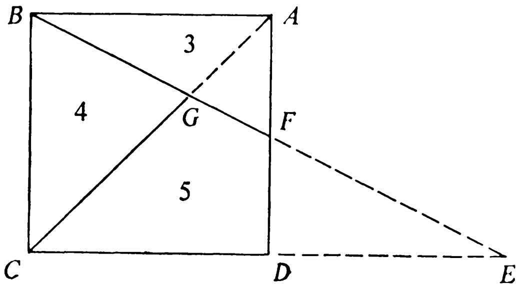

FIG. 14

The alert reader may find 3:4:5 triangles imbedded in many other configurations. The authors will be appreciative of any such discoveries that are brought to their attention.

References

1. C. W. Trigg and M. W. Fleck, E 432, Amer. Math. Monthly, 48 (1941) 267–268.
2. John Casey, A Sequel to Euclid, Longmans, Green, and Co., London, 1884, p. 118, Prop. 7.
3. R. A. Johnson, Modern Geometry, Houghton Mifflin, Boston, 1929, p. 113.
4. Leon Bankoff and W. J. Cherry, A 3 :4 :5 right triangle, Amer. Math. Monthly, 61 (1954) 473–474.
5. Leon Bankoff and W. J. Blundon, An infinite sequence of Pythagorean triangles, Amer. Math. Monthly, 62 (1955) 734–736.
6. C. W. Trigg, Configuration generated by folding a square, Scripta Math., 21 (1955) 77–80.
7. Michael Goldberg and Aaron Buchman, A dissection of a triangle, Amer. Math. Monthly, 58 (1951) 112.
8. C. W. Trigg, Mathematical Quickies, McGraw-Hill, New York, 1967, p. 140.
9. Leon Bankoff and D. M. Brown, A square dissection, this MAGAZINE, 29 (1955) 110–112.

PROOFS OF URYSOHN'S LEMMA AND RELATED THEOREMS BY MEANS OF ZORN'S LEMMA

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1. Introduction. Urysohn's Lemma [**10**, § 25] is the assertion that if A and B are disjoint closed subsets of a normal topological space X, then there exists a real-valued continuous function f on X such that $f(x) = 0$ for every $x \in A$ and $f(x) = 1$ for every $x \in B$. The standard proofs of this fundamental result (see, e.g., Dugundji [**3**], Chap. VII, Theorem 4.1 or Gillman and Jerison [**5**], 3.13) involve three main steps: (1) construction, by recursion (or, as most authors say, by "induction"), of a certain family of open subsets of X, (2) verification, by induction, that this family has certain properties (involving the closures of the members of the family), and (3) construction of the desired continuous function f from this family of open sets.

Another basic theorem of general topology is Urysohn's Extension Theorem [**10**, §28]: Every bounded real-valued continuous function on a closed subset of a normal space X can be extended continuously over X. (The special case of this result, with "normal" replaced by "metric", is due to Tietze [**9**, Satz 3].) Once again, the standard proof (see, e.g., Dugundji [**3**], Chap. VII, Theorem 5.1) involves essentially the same three steps mentioned above (with "family of open subsets of X" replaced by "sequence of continuous functions on X").

Now an inspection of the proofs of these two results shows that step (1) actually involves an application of the Axiom of Choice. To the best of the author's knowledge, however, this fact is never made explicit in any of the standard expositions of the fundamentals of general topology (thus leaving, perhaps, a gap in the student's understanding at a crucial point of the proofs). At the same time, step (2) is virtually ignored. It seems worthwhile, therefore, to provide proofs of Urysohn's Lemma and of Urysohn's Extension Theorem that are completely free of these defects. We shall do this here by replacing, in the proof of each of these theorems, steps (1) and (2) by a single application of Zorn's Lemma (which, of course, is equivalent to the Axiom of Choice). It also seems appropriate to include a similar Zorn's Lemma proof of a third important (but somewhat more technical) fact about normal spaces, namely, that every point-finite open cover of a normal space has an "open shrinkage" (Lefschetz [**8**], Chap. I, 33.4, Dieudonné [**2**], Théorème 6). (There are other Zorn's Lemma proofs of this result in the literature ([**2**], [**4**], Theorem 20.10, [**7**], 13.14, [**8**]), but perhaps the more common proof relies on the well-ordering theorem instead of on Zorn's Lemma (see, e.g., Dugundji [**3**], Chap. VII, Theorem 6.1 or Bourbaki [**1**], Chap. IX, §4, Theorem 3).) Our approach will thus lend a certain unity (and clarity) to the proofs of these three basic theorems concerning normal spaces, and, at the same time, will provide illustrations of three distinctly nontrivial applications of Zorn's Lemma.

We shall denote the natural numbers, the rational numbers, and the real numbers by **N**, **Q**, and **R**, respectively. If X is a topological space and if $S \subset X$, then the closure of S in X will be denoted by cl S or by $\text{cl}_X S$. The set of all real-valued continuous functions (resp. all bounded real-valued continuous functions) on X will be denoted by $C(X)$ (resp. $C^*(X)$). (The reader is assumed to be familiar with the usual algebraic operations in $C(X)$, including the lattice operations of meet $\wedge$ and join $\vee$ (see, e.g., Gillman and Jerison [**5**], pp. 10–11).) If $r \in \mathbf{R}$, then we denote by $\boldsymbol{r}$ the constant function on X whose constant value is r. Moreover, if $f \in C(X)$, if $S \subset X$, and if $r \in \mathbf{R}$, then the statement "$f = \boldsymbol{r}$ on S" will mean that $f(x) = r$ for every $x \in S$.

An attempt has been made to make this paper reasonably self-contained. For notation and terminology not defined here, however, and for general background, the reader may consult [**1**], [**3**], [**5**], and [**6**].

2. Urysohn's Lemma. Urysohn's Lemma constructs a continuous function in a context where no functions are given in advance. This is possible because what we shall call a "Urysohn family" in X implies the existence of a certain function $f \in C(X)$; and the context of Urysohn's Lemma (namely, *normality* of the space X) is precisely that in which the existence of appropriate Urysohn families can be proved. We begin, therefore, with the definition of a Urysohn family:

DEFINITION. *By a Urysohn family in X we mean a family $(U_r)_{r \in \mathbf{Q}}$ of open sets in X that is indexed by the rational numbers $\mathbf{Q}$ and that satisfies the following three conditions:*

(1) $\bigcup_{r \in \mathbf{Q}} U_r = X$.

(2) $\bigcap_{r \in \mathbf{Q}} U_r = \varnothing$.

(3) $\text{cl}\, U_r \subset U_s$ *whenever* $r < s$.

If $\mathcal{U} = (U_r)_{r \in \mathbf{Q}}$ is a Urysohn family in X and if $x \in X$, then the preceding definition implies that $\{r \in \mathbf{Q} : x \in U_r\}$ is a nonempty subset of $\mathbf{R}$ that is bounded below. We may therefore define a real-valued function $\Phi(\mathcal{U})$ on X as follows: For each $x \in X$,

$$\Phi(\mathcal{U})(x) = \inf\{r \in \mathbf{Q} : x \in U_r\}.$$

LEMMA 2.1. *Let $\mathcal{U} = (U_r)_{r \in \mathbf{Q}}$ be a Urysohn family in X and let $f = \Phi(\mathcal{U})$. If $s \in \mathbf{Q}$ and if $f(x) < s$, then $x \in U_s$.*

Proof. Since $f(x) < s$, the definition of f implies that there exists $r \in \mathbf{Q}$ such that $x \in U_r$ and $r < s$. But then $\text{cl}\, U_r \subset U_s$ and hence $x \in U_s$.

LEMMA 2.2. *If $\mathcal{U} = (U_r)_{r \in \mathbf{Q}}$ is a Urysohn family in X and if $f = \Phi(\mathcal{U})$, then $f \in C(X)$.*

Proof. Let $x \in X$, let (a, b) be an open interval of $\mathbf{R}$ that contains $f(x)$, choose $r, s, t \in \mathbf{Q}$ such that

$$a < r < t < f(x) < s < b,$$

and let

$$V = U_s - \text{cl}\, U .$$

By Lemma 2.1, $x \in U_s$. Moreover, $x \notin U_t$ and cl $U_r \subset U_t$, so $x \notin$ cl U_r. Thus the open set V is a neighborhood of x. Now for each $y \in V$, we have $r \leqq f(y) \leqq s$ (by Lemma 2.1 and the definition of f) and thus $f(V) \subset (a, b)$. It follows that f is continuous, and the proof is complete.

Two subsets A and B of X are said to be *completely separated* in X in case there exists a function $f \in C(X)$ such that $f = \mathbf{0}$ on A, $f = \mathbf{1}$ on B, and $\mathbf{0} \leqq f \leqq \mathbf{1}$. Note that if $r, s \in \mathbf{R}$ with $r < s$, and if f has the properties just described, then there exists a function $g \in C(X)$ such that $g = \mathbf{r}$ on A, $g = \mathbf{s}$ on B, and $\mathbf{r} \leqq g \leqq \mathbf{s}$ (simply take $g = f(\mathbf{s} - \mathbf{r}) + \mathbf{r}$). On the other hand, if there exists a function $g \in C(X)$ such that $g(x) \leqq 0$ for every $x \in A$ and $g(x) \geqq 1$ for every $x \in B$, then A and B are completely separated. (The function $f = (g \vee \mathbf{0}) \wedge \mathbf{1}$ is in $C(X)$ and we have $f = \mathbf{0}$ on A, $f = \mathbf{1}$ on B, and $\mathbf{0} \leqq f \leqq \mathbf{1}$.)

LEMMA 2.3. *If A and B are subsets of X, then the following statements are equivalent:*

(i) *A and B are completely separated in X.*

(ii) *There exists a Urysohn family $\mathscr{U} = (U_r)_{r \in \mathbf{Q}}$ in X such that $A \subset U_0$ and $B \subset X - U_1$.*

Proof. (i) $\Rightarrow$ (ii). By hypothesis, there exists $f \in C(X)$ such that $f = \mathbf{0}$ on A and $f = \mathbf{1}$ on B. If, for each $r \in \mathbf{Q}$, we set

$$U_r = \{x \in X : f(x) < (r + 1)/2\},$$

then $(U_r)_{r \in \mathbf{Q}}$ is a Urysohn family in X such that $A \subset U_0$ and $B \subset X - U_1$.

(ii) $\Rightarrow$ (i). Set $f = \Phi(\mathscr{U})$ and note that $f \in C(X)$ by Lemma 2.2. Since $A \subset U_0$, we have $f(x) \leqq 0$ for every $x \in A$. Moreover, for every $x \in B$ we have $x \notin U_1$, and hence $f(x) \geqq 1$ by Lemma 2.1. It follows that A and B are completely separated in X.

The usual proof of Urysohn's Lemma involves the use of dyadic numbers (see, e.g., Dugundji [**3**], Chap. VII, Theorem 4.1). In contrast, the elegant argument of Gillman and Jerison [**5**, 3.13] completely avoids their use. It is this latter argument that we shall adapt in the following proof of Urysohn's Lemma.

URYSOHN'S LEMMA. *If X is a normal space, then any two disjoint closed subsets of X are completely separated in X.*

Proof. Let A and B be disjoint closed subsets of X. By normality, there exists an open set U such that

$$A \subset U \subset \text{cl } U \subset X - B.$$

Let $n \to r_n$ be an arbitrary bijection from $\mathbf{N}$ onto $\mathbf{Q} \cap [0, 1]$ (where $[0, 1]$ is the closed unit interval) such that $r_0 = 1$ and $r_1 = 0$, and let $\mathfrak{P}$ be the set of all sequences $(G_n)_{n \in J}$ of open subsets of X that satisfy the following three conditions:

(a) $\{0, 1\} \subset J \subset \mathbf{N}$, $G_0 = X - B$, and $G_1 = U$.

(b) If $j \in J$ and if $i \in \mathbf{N}$ such that $i \leqq j$, then $i \in J$.

(c) If $i, j \in J$ and if $r_i < r_j$, then cl $G_i \subset G_j$.

Partially order $\mathfrak{P}$ as follows: $(G_n)_{n \in J} \leqq (H_n)_{n \in K}$ in case $(H_n)_{n \in K}$ is an *extension* of $(G_n)_{n \in J}$ (i.e., $J \subset K$ and $G_n = H_n$ for every $n \in J$). Note that $\mathfrak{P}$ is nonempty since the sequence $(G_n)_{n \in \{0,1\}}$ belongs to $\mathfrak{P}$, where $G_0 = X - B$ and $G_1 = U$.

Consider any nonempty totally ordered family

$$\Lambda = ((G_{\lambda n})_{n \in J_\lambda})_{\lambda \in L}$$

of elements of $\mathfrak{P}$ and let $J = \bigcup_{\lambda \in L} J_\lambda$. Since Λ is totally ordered, there clearly exists a family $\mathscr{G} = (G_n)_{n \in J}$ of open sets that is a common extension of every $(G_{\lambda n})_{n \in J_\lambda}$, and (since Λ is totally ordered and nonempty) it is also clear that $\mathscr{G} \in \mathfrak{P}$. Thus $\mathscr{G}$ is an upper bound for Λ in $\mathfrak{P}$, and it follows by Zorn's Lemma that the partially ordered set $\mathfrak{P}$ has a maximal element $\mathscr{H} = (H_n)_{n \in M}$.

We assert that $M = \mathbf{N}$. If this is not the case, then the set $\mathbf{N} - M$ has a least element m (and M is finite since $\mathscr{H}$ satisfies (b)). Let

$$P = \{r_n : n \in M \text{ and } r_n < r_m\}$$

and

$$Q = \{r_n : n \in M \text{ and } r_m < r_n\},$$

and note that both P and Q are finite and nonempty (since $r_1 < r_m < r_0$). It follows that P (resp. Q) has a largest (resp. smallest) element r_p (resp. r_q). Since $r_p < r_q$, we have cl $H_p \subset H_q$, and therefore, by normality again, there exists an open set V such that

$$\text{cl } H_p \subset V \subset \text{cl } V \subset H_q.$$

If we set $W_n = H_n$ for every $n \in M$ and $W_m = V$, then we claim that $\mathscr{W} = (W_n)_{n \in M \cup \{m\}}$ belongs to $\mathfrak{P}$. Clearly $\mathscr{W}$ satisfies (a), and $\mathscr{W}$ satisfies (b) by the minimality of m To verify (c), let $i, j \in M \cup \{m\}$, assume that $r_i < r_j$, and note that, so far as (c) is concerned, we may assume that either $i = m$ or $j = m$. If $j = m$, then necessarily $i \in M$, whence (by the maximality of r_p) $r_i \leqq r_p$. But then

$$\text{cl } W_i = \text{cl} H_i \subset \text{cl } H_p \subset V = W_j.$$

Similarly, if $i = m$, then the minimality of r_q insures that cl $W_i \subset W_j$. Thus $\mathscr{W} \in \mathfrak{P}$, contrary to the maximality of $\mathscr{H}$, and we conclude that $M = \mathbf{N}$.

Now, for each $r \in \mathbf{Q}$, set $U_r = \varnothing$ if $r < 0$, $U_r = X$ if $r > 1$, and $U_r = H_n$ if $r = r_n$ for some $n \in \mathbf{N}$. Then clearly $(U_r)_{r \in \mathbf{Q}}$ is a Urysohn family in X, and we have $A \subset U_0$ and $B = X - U_1$. It follows from Lemma 2.3 that A and B are completely separated, and the proof is complete.

3. Urysohn's Extension Theorem. A subset S of a topological space X is said to be C^*-*embedded* in X in case every function in $C^*(S)$ can be extended to a function in $C^*(X)$. We shall prove below the generalized version of Urysohn's Extension Theorem given by Gillman and Jerison in [5, Theorem 1.17]. This important result provides a necessary and sufficient condition for a subset of X to be C^*-embedded in X. As a corollary one obtains, with the aid of Urysohn's Lemma, the classical form of Urysohn's Extension Theorem quoted in the introduction. We begin with a definition and two lemmas:

DEFINITION. *If $f \in C(X)$, then by the zero-set of f we mean the set $\mathbf{Z}(f) = \{x \in X : f(x) = 0\}$. A subset A of X is a zero-set in X in case $A = \mathbf{Z}(f)$ for some $f \in C(X)$.*

LEMMA 3.1. *If $f \in C(X)$ and if $r \in \mathbf{R}$, then the sets $\{x \in X : f(x) \leqq r\}$ and $\{x \in X : f(x) \geqq r\}$ are zero-sets in X.*

Proof. We need only note that

$$\{x \in X : f(x) \leqq r\} = \mathbf{Z}((f \vee \boldsymbol{r}) - \boldsymbol{r})$$

and

$$\{x \in X : f(x) \geqq r\} = \mathbf{Z}((f \wedge \boldsymbol{r}) - \boldsymbol{r}).$$

LEMMA 3.2 [**5**, Theorem 1.15]. *If A and B are disjoint zero-sets in X, then A and B are completely separated in X.*

Proof. We have $A = \mathbf{Z}(f)$ and $B = \mathbf{Z}(g)$ for some f, $g \in C(X)$. Since $\mathbf{Z}(f) \cap \mathbf{Z}(g) = \varnothing$, the function $|f| + |g|$ never vanishes, and we may therefore define a function $h \in C(X)$ by the formula $h = |f| \cdot (|f| + |g|)^{-1}$. Clearly $h = \mathbf{0}$ on A, $h = \mathbf{1}$ on B, and $\mathbf{0} \leqq h \leqq \mathbf{1}$.

URYSOHN'S EXTENSION THEOREM. *If S is a subset of a topological space X, then the following statements are equivalent:*

(i) *S is C^*-embedded in X.*

(ii) *If A and B are any two subsets of S that are completely separated in S, then A and B are also completely separated in X.*

Proof. (i) $\Rightarrow$ (ii). This is the easy implication. If A and B are completely separated in S, then there exists a function $f \in C^*(S)$ such that $f = \mathbf{0}$ on A and $f = \mathbf{1}$ on B. Since S is C^*-embedded in X, we can extend f to a function $g \in C^*(X)$. Obviously $g = \mathbf{0}$ on A and $g = \mathbf{1}$ on B, so A and B are completely separated in X.

(ii) $\Rightarrow$ (i). Consider any function $f \in C^*(S)$. Our problem is to extend f to a function in $C^*(X)$. Since f is bounded, we have $|f| \leqq \boldsymbol{p}$ for some positive integer p. For each $n \in \mathbf{N}$, let

$$r_n = \frac{p}{2}\left(\frac{2}{3}\right)^n,$$

and let $\mathfrak{P}$ be the set of all sequences $(f_n)_{n \in J}$ of functions in $C(X)$ that satisfy the following three conditions:

(a) $0 \in J \subset \mathbf{N}$ and $f_0 = 0$.

(b) If $j \in J$ and if $i \in \mathbf{N}$ such that $i \leqq j$, then $i \in J$.

(c) For each $n \in J$, $|f_n| \leqq \boldsymbol{r}_n$ and $|f - \sum_{i=0}^{n} (f_i | S)| \leqq 3\boldsymbol{r}_{n+1}$.

Once again, partially order $\mathfrak{P}$ by extension: $(f_n)_{n \in J} \leqq (g_n)_{n \in K}$ in case $J \subset K$ and $f_n = g_n$ for every $n \in J$. Note that $\mathfrak{P}$ is nonempty since the sequence $(f_n)_{n \in \{0\}}$ belongs to $\mathfrak{P}$, where $f_0 = \mathbf{0}$.

Now, just as in the proof of Urysohn's Lemma, one easily verifies that every nonempty totally ordered family of elements of $\mathfrak{P}$ has an upper bound in $\mathfrak{P}$. Zorn's Lemma therefore yields a maximal element $\mathscr{G} = (g_n)_{n \in M}$ of the partially ordered set $\mathfrak{P}$. We shall show that $M = \mathbf{N}$ and that the desired extension of f can be taken to be the sum of the series $\Sigma_{n \in N} g_n$.

Suppose that $M \neq \mathbf{N}$ so that $\mathbf{N} - M$ has a least element m (whence $m - 1 \in M$). Let

$$A = \left\{ x \in S : f(x) - \sum_{i=0}^{m-1} g_i(x) \leqq -r_m \right\}$$

and

$$B = \left\{ x \in S : f(x) - \sum_{i=0}^{m-1} g_i(x) \geqq r_m \right\}.$$

By Lemma 3.1, the sets A and B are (disjoint) zero-sets in S, and therefore, by Lemma 3.2, A and B are completely separated in S. By hypothesis, then, A and B are also completely separated in X, so there exists a function $h \in C(X)$ such that $h = -\mathbf{r}_m$ on A, $h = \mathbf{r}_m$ on B, and $|h| \leqq \mathbf{r}_m$. If we set $h_n = g_n$ for every $n \in M$ and $h_m = h$, then we claim that $\mathscr{H} = (h_n)_{n \in M \cup \{m\}}$ belongs to $\mathfrak{P}$. Clearly $\mathscr{H}$ satisfies (a), and $\mathscr{H}$ satisfies (b) by the minimality of m. To verify that $\mathscr{H}$ satisfies (c), consider any $n \in M \cup \{m\}$ and observe that, so far as (c) is concerned, we may assume that $n = m$ (whence $|h_n| \leqq \mathbf{r}_n$). Let $k = f - \Sigma_{i=0}^{m-1}(g_i | S)$, and note that the values of h and k lie between $-3r_m$ and $-r_m$ on A, between r_m and $3r_m$ on B, and between $-r_m$ and r_m on $S - (A \cup B)$. It follows that

$$\left| f - \sum_{i=0}^{m} (h_i | S) \right| = |k - (h | S)| \leqq 2\mathbf{r}_m \leqq 3\mathbf{r}_{m+1},$$

and we have established (c). Thus $\mathscr{H} \in \mathfrak{P}$, contrary to the maximality of $\mathscr{G}$, and we conclude that $M = \mathbf{N}$.

Now since $|g_n| \leqq \mathbf{r}_n$ for every $n \in \mathbf{N}$, the Weierstrass M-test (see, e.g., [3], Chap. III, 10.5) implies that if the function g is defined by the formula

$$g(x) = \Sigma_{n \in N} g_n(x) \qquad (x \in X),$$

then $g \in C^*(X)$. Moreover, since $(g_n)_{n \in N}$ satisfies condition (c), it follows that for each $x \in S$ we have $\Sigma_{n \in N} g_n(x) = f(x)$. Thus g is an extension of f, so we have proved that S is C^*-embedded in X. The proof is now complete.

Urysohn's original extension theorem for normal spaces is easily deducible from the preceding theorem:

COROLLARY. *Every closed subset of a normal space X is C^*-embedded in X.*

Proof. Let F be a closed subset of X and suppose that A and B are subsets of F that are completely separated in F. Then there exists a function $f \in C(F)$ such that $f = \mathbf{0}$ on A and $f = \mathbf{1}$ on B. Since f is continuous, it follows that $f = \mathbf{0}$ on $\mathrm{cl}_F A$ and $f = \mathbf{1}$ on $\mathrm{cl}_F B$. The sets $\mathrm{cl}_F A$ and $\mathrm{cl}_F B$ are therefore disjoint, and obviously they are closed in X. Urysohn's Lemma then implies that $\text{cl}_F A$ and $\text{cl}_F B$, and hence also A and B, are completely separated in X. The result now follows from the preceding theorem.

REMARK. A result considerably stronger than the preceding corollary is actually true: *Every closed subset of a normal space X is C-embedded in X.* (A subset S of a space X is *C-embedded* in X in case every function in $C(S)$ can be extended to a function in $C(X)$.) This is an immediate consequence of the above corollary, Urysohn's Lemma, the fact that zero-sets are always closed, and the following theorem of Gillman and Jerison [5, Theorem 1.18]: *A C^*-embedded subset of a space X is C-embedded in X if and only if it is completely separated from every zero-set disjoint from it.* We shall not prove this latter result here.

4. Shrinkages of open covers. A family $(A_\alpha)_{\alpha \in I}$ of subsets of a space X is said to be *point-finite* in case, for each $x \in X$, there exist at most finitely many indices $\alpha \in I$ such that $x \in A_\alpha$. If $(A_\alpha)_{\alpha \in I}$ is a cover of X, then by an *open shrinkage* of $(A_\alpha)_{\alpha \in I}$ we mean an open cover $(B_\alpha)_{\alpha \in I}$ of X such that $\text{cl } B_\alpha \subset A_\alpha$ for every $\alpha \in I$.

If every point-finite open cover of X has an open shrinkage, then it is easy to prove that X is normal (see, e.g., the proof of [3], Chap. VII, Theorem 6.1). We shall now prove the nontrivial converse of this assertion:

THEOREM. *If X is normal, then every point-finite open cover of X has an open shrinkage.*

Proof. Let $\mathscr{G} = (G_\alpha)_{\alpha \in I}$ be a point-finite open cover of X and let $\mathfrak{P}$ be the set of all families $(U_\alpha)_{\alpha \in J}$ of open subsets of X that satisfy the following three conditions:

(a) $J \subset I$.

(b) For each $\alpha \in J$, $\text{cl } U_\alpha \subset G_\alpha$.

(c) $X = (\bigcup_{\alpha \in J} U_\alpha) \cup (\bigcup_{\alpha \in I - J} G_\alpha)$.

Partially order $\mathfrak{P}$ by extension and note that $\mathfrak{P}$ is nonempty since the family $(U_\alpha)_{\alpha \in \varnothing}$ belongs to $\mathfrak{P}$.

Consider any totally ordered family

$$\Lambda = ((U_{\lambda\alpha})_{\alpha \in J_\lambda})_{\lambda \in L}$$

of elements of $\mathfrak{P}$ and let $J = \bigcup_{\lambda \in L} J_\lambda$. Since Λ is totally ordered, there exists a family $\mathscr{U} = (U_\alpha)_{\alpha \in J}$ that is a common extension of every $(U_{\lambda\alpha})_{\alpha \in J_\lambda}$, and clearly $\mathscr{U}$ satisfies (a) and (b). To verify that $\mathscr{U}$ satisfies (c), let $x \in X$, suppose that $x \notin \bigcup_{\alpha \in I - J} G_\alpha$, and let

$$K = \{\alpha \in I : x \in G_\alpha\}.$$

Then $K \subset J$ and, since $(G_\alpha)_{\alpha \in I}$ is point-finite, K is finite. Since Λ is totally ordered, it follows that $K \subset J_\lambda$ for some $\lambda \in L$. Then $x \notin \bigcup_{\alpha \in I - J_\lambda} G_\alpha$, so

$$x \in \bigcup_{\alpha \in J_\lambda} U_{\lambda\alpha} = \bigcup_{\alpha \in J_\lambda} U_\alpha \subset \bigcup_{\alpha \in J} U_\alpha,$$

and we conclude that $\mathscr{U}$ satisfies (c). Thus $\mathscr{U}$ is an upper bound for Λ in the partially ordered set $\mathfrak{P}$, and it follows by Zorn's Lemma that $\mathfrak{P}$ has a maximal element $\mathscr{H} = (H_\alpha)_{\alpha \in M}$. To complete the proof, it will obviously suffice to show that $M = I$.

Suppose, on the contrary, that there exists an element $\beta \in I - M$ and let $J = M \cup \{\beta\}$. Since $\mathscr{H}$ satisfies (c), it follows that

$$X - G_\beta \subset (\bigcup_{\alpha \in M} H_\alpha) \cup (\bigcup_{\alpha \in I-J} G_\alpha),$$

and therefore, since X is normal, there exists an open subset V of X such that

$$X - G_\beta \subset V \subset \mathrm{cl}\, V \subset (\bigcup_{\alpha \in M} H_\alpha) \cup (\bigcup_{\alpha \in I-J} G_\alpha).$$

If we set $V_\alpha = H_\alpha$ for every $\alpha \in M$ and $V_\beta = X - \mathrm{cl}\, V$, then we claim that $\mathscr{V} = (V_\alpha)_{\alpha \in J}$ belongs to $\mathfrak{P}$. Clearly $\mathscr{V}$ satisfies (a), and $\mathscr{V}$ satisfies (b) since $\mathscr{H}$ satisfies (b) and

$$\mathrm{cl}\, V_\beta = \mathrm{cl}(X - \mathrm{cl}\, V) \subset X - V \subset G_\beta.$$

To verify that $\mathscr{V}$ satisfies (c), let $x \in X$ and suppose that $x \notin \bigcup_{\alpha \in J} V_\alpha$. Then $x \in \mathrm{cl}\, V$ and $x \notin \bigcup_{\alpha \in M} H_\alpha$, whence $x \in \bigcup_{\alpha \in I-J} G_\alpha$. Thus $\mathscr{V}$ satisfies (c), so $\mathscr{V} \in \mathfrak{P}$, contrary to the maximality of $\mathscr{H}$. We conclude that $M = I$, and the proof is complete.

REMARK. The preceding theorem has a number of applications. For example, it (together with Urysohn's Lemma) is one of the basic ingredients of the proof of the following important fact (see, e.g., Dugundji [3], Chap. VIII, Theorem 4.2): *Every locally finite open cover of a paracompact space has a partition of unity subordinate to it.*

References

1. N. Bourbaki, Elements of Mathematics: General Topology, Parts 1 and 2, Hermann, Paris; Addison-Wesley, Reading, Mass., 1966.

2. J. Dieudonné, Une généralisation des espaces compacts, J. Math. Pures Appl., 23 (1944) 65–76.

3. J. Dugundji, Topology, Allyn and Bacon, Boston, 1965.

4. W. W. Fairchild and C. Ionescu Tulcea, Topology, W. B. Saunders, Philadelphia, London, Toronto, 1971.

5. L. Gillman and M. Jerison, Rings of Continuous Functions, Van Nostrand, Princeton, 1960.

6. J. L. Kelley, General Topology, Van Nostrand, New York, 1955.

7. H. J. Kowalsky, Topological Spaces, Academic Press, New York, 1965.

8. S. Lefschetz, Algebraic Topology, Amer. Math. Soc. Colloq. Publications, vol. 27, New York, 1942.

→ H. Tietze, Über Funktionen die auf einer abgeschlossenen Menge stetig sind, J. Reine Angew. Math., 145 (1915) 9–14.

→ P. Urysohn, Über die Mächtigkeit der zusammenhängenden Mengen, Math. Ann., 94 (1925) 262–295.

SOME EXTREMAL PROBLEMS IN ELEMENTARY PROBABILITY THEORY

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1. Introduction. Suppose we are given a fixed collection of m balls colored with k colors $c_1, \cdots, c_k$, so that there are m_i balls of color c_i, $i = 1, \cdots, k$, where $0 < m_i < m$, and $\sum_{i=1}^{k} m_i = m$. By a *configuration* C we shall mean a finite collection of indistinguishable boxes, together with an assignment of the given m balls into the boxes. It should be emphasized that we have *not* preassigned any restriction on the number of boxes allowed in a configuration. Furthermore, we assume that a given box in a configuration can contain anywhere from 0 to all m balls. Hence, if C has q boxes, $B_1, \cdots, B_q$, then the assignment of the balls can be specified by nonnegative integers $m(i,j)$, $i = 1, \cdots, k$, $j = 1, \cdots, q$, where $m(i,j)$ denotes the number of balls of color c_i in the box B_j, so that $0 \leqq m(i,j) \leqq m_i$, $0 \leqq \sum_{i=1}^{k} m(i,j) \leqq m$, and $\sum_{j=1}^{q} m(i,j) = m_i$. When classifying configurations, we disregard any ordering of the boxes in a configuration, so that any configuration obtained from C by a renumbering of the boxes in C is regarded as being equivalent to C.

Given any configuration C, consider the following *experiment*: a box in C is selected at random, and if the selected box is not empty, then a ball is drawn at random from the box. We shall concern ourselves in this paper with the problem of *finding configurations which maximize, amongst all configurations containing the given m balls, the probability of obtaining a given predesignated outcome after the repetition of the above experiment a specified number of times*, (*thereby also determining this maximal probability*).

By way of illustration, suppose that we are given three red balls and two white balls, and we wish to find a configuration which maximizes, amongst all configurations containing these balls, the probability of obtaining a red ball upon a single performance of the experiment. The (unique) solution to this problem is easily seen to be a configuration with three boxes, where single red balls are placed in each of two of the boxes, and the remaining red ball and both white balls are put in the third box, yielding a probability of 7/9 that a red ball is drawn. As a second illustration, if we are given three red balls and one white ball, then there are two different configurations which maximize the probability of obtaining a red ball upon a single performance of the experiment: one configuration has two boxes, with a single red ball in one box and the remaining two red balls and the white ball in the other box, whereas a second configuration has three boxes, with single red balls in each of two of the boxes, and the remaining red ball and the white ball in the third box, yielding a probability of 5/6 that a red ball is drawn for either configuration. These examples are special cases of the general "single draw problem", whose complete solution is presented in Section 2.

If we are performing the experiment more than once, we shall assume that the balls are replaced after each draw, (sampling with replacement), so that a configuration remains unchanged for each repetition. To illustrate, suppose that we are given three red balls, two white balls, and two green balls, and we wish to find a configuration which maximizes the probability of obtaining a red ball and a white ball, (in either order), after performing the experiment twice. The (unique) solution to this problem is a configuration having four boxes, with single white balls in each of two of the boxes, a single red ball in the third box, and the remaining two red balls and the two green balls in the fourth box. Note that this example is analogous to the single draw case in that not every color is to be drawn. By way of contrast, to illustrate the case where at least one ball of every color is to be drawn, suppose we are given three red balls and two white balls, and we wish to find a configuration which maximizes the probability of obtaining a red ball and a white ball, (in either order), after performing the experiment twice. Letting $p(C, c_i)$ denote the probability, for a configuration C, that a ball of color c_i is obtained at a given performance of the experiment, it is easily seen that any configuration C such that $p(C, \text{red}) = p(C, \text{white}) = \frac{1}{2}$ is a solution to the problem. There are three nonequivalent configurations which do the job. This last example is a special case of a partial solution (Theorem 1) presented in Section 3 to the "repetition with replacement problem" for those events in which, unlike the single draw problem, at least one ball of every color is to be drawn at some performance of the experiment.

2. The single draw problem. In analyzing the single draw problem, it clearly suffices to assume that we only have two colors, say red and white. Let R = number of red balls, and W = number of white balls, in our given collection. Since there is obviously no solution to the problem of finding a configuration which maximizes the selection of an empty box, we can disregard this case. Then, by symmetry, we need only consider the problem of finding a configuration which maximizes the probability of obtaining a red ball. Let $C^* = C^*(R, W)$ denote, throughout the remainder of this section, such a "maximal" configuration, (C^* clearly exists, but it might not be unique). The following four statements will lead to a determination of C^*:

(1) C^* has no empty boxes,

(2) a box in C^* which does not contain a white ball contains only one red ball,

(3) all the white balls are in one box,

(4) there are at most $R + 1$ boxes.

Statement (1) is obvious, and statement (4) follows immediately from (1) and (3). If a configuration does *not* satisfy (2), then it is easily checked that adding a box and transferring a red ball from a box violating (2) to the added box increases the probability of drawing a red ball. Finally, if a configuration contains two boxes containing white balls, the inequality

$$\frac{r_1}{r_1 + w_1} + \frac{r_2}{r_2 + w_2} < 1 + \frac{r_2}{r_2 + w_1 + w_2}, \qquad (r_1 \leqq r_2;\ r_1, r_2, w_1, w_2 > 0),$$

shows that the probability of drawing a red ball is increased by transferring all of the white balls in one of the two boxes to the other box, thereby verifying (3). If a maximal configuration C^* has q boxes, then (1)–(4) say that the (maximal) probability of drawing a red ball, denoted by $p(C^*, \text{red})$, is given by

$$p(C^*, \text{red}) = \frac{1}{q}\left(q - 1 + \frac{R - q + 1}{R - q + 1 + W}\right)$$

$$= 1 - p(C^*, \text{white}) = 1 - \frac{W}{q(R - q + 1 + W)},$$

with the arrangement of the balls being dictated by (1)–(3). Hence, the problem of determining C^* reduces to finding the maximum value of $1 - W/q(R - q + 1 + W)$ for $q \in \{1, 2, \cdots, R + 1\}$. Now the second degree polynomial $f(x) = x(R - x + 1 + W)$ has an absolute maximum when $x = \frac{1}{2}(R + W + 1)$. Therefore, the maximum of f on the interval $[1, R + 1]$ occurs at the right hand endpoint $R + 1$ if $W \geqq R + 1$, whereas if $W \leqq R$, then the maximum of f occurs at the interior point $\frac{1}{2}(R + W + 1)$ of the interval. The following description of the number of boxes q in C^* has therefore been established:

(i) if $W \geqq R + 1$, then $q = R + 1$,

(ii) if $W \leqq R$, then $q = \frac{1}{2}(R + W + 1)$ when $R + W$ is odd, and $q = \frac{1}{2}(R + W)$ or $\frac{1}{2}(R + W) + 1$ when $R + W$ is even.

Since, as we have seen, q completely determines C^*, the single draw problem is completely solved.

3. The multiple repetition with replacement problem. Given a configuration C containing the fixed collection of m balls, suppose we perform the experiment discussed in Section 1 n times in succession, $n \geqq 2$, where we assume that the balls are replaced after each repetition. Since an ordering is necessarily implicit in this process, we obtain n ordered outcomes after repeating the experiment n times. Such an ordered succession of outcomes will be called an *ordered* $(C, n, +)$-*event*, where the $+$ indicates that we are sampling with replacement. An ordered $(C, n, +)$-event can be specified by, and identified with, an n-tuple $(i_1, \cdots, i_n)$, where c_{i_j} is the color of the ball drawn at the jth repetition, and where we set $i_j = 0$ in the case an empty box is selected.

If we choose to disregard the ordering in our n repetitions of the experiment, and merely total up the number of times a ball of color c_j was drawn, $j = 1, \cdots, k$, then we are led to define an equivalence relation in the set of ordered $(C, n, +)$-events by

$$(i_1, \cdots, i_n) \sim (j_1, \cdots, j_n)$$

if there is a permutation π of $\{1, 2, \cdots, n\}$ such that

$$j_q = i_{\pi(q)}, \; q = 1, 2, \cdots, n.$$

Equivalence classes relative to $\sim$ will be called $(C, n, +)$-events. Note that a $(C, n, +)$-event Q determines, and can be identified with, a (unique) $(k + 1)$-tuple $\alpha = (n_0, n_1, \cdots, n_k)$, where n_j is the number of occurrences of i_j in any representative of Q. By abuse of language, we say that $(\alpha, +)$ *represents* Q. The $(C, n, +)$-event Q will be called *nondegenerate* if $n_0 = 0$ where $(\alpha, +) = ((n_0, n_1, \cdots, n_k), +)$ represents Q, otherwise Q is called *degenerate*. We now formally state the multiple repetition with replacement problem, recalling that we are given a fixed collection of m balls of colors $c_1, \cdots, c_k$, so that there are $m_i > 0$ balls of color c_i, and $\sum_{i=1}^{k} m_i = m$.

PROBLEM. *Given* $\alpha = (n_0, n_1, \cdots, n_k)$ *such that* $0 \leqq n_j \leqq n$, $\sum_{j=0}^{k} n_j = n \geqq 2$, *and* $n_0 < n$, *find a configuration* $C^* = C^*(\alpha, +)$ (*it might not be unique*) *which maximizes, amongst all configurations C containing the given m balls, the probability of the* $(C, n, +)$-*event represented by* $(\alpha, +)$, *thereby also determining the value of this maximal probability.*

A configuration $C^* = C^*(\alpha, +)$ which solves the above problem is called *maximal* for $(\alpha, +)$, where we have assumed $n_0 < n$ to insure that C^* exists. Given a configuration C, let $p(C, \alpha, +)$ denote the probability of the $(C, n, +)$-event represented by $(\alpha, +)$. Let $p_i = p_i(C, c_i)$ denote the probability that a ball of color c_i is drawn at a given repetition of the experiment, $i = 1, \cdots, k$, and let $p_0 = p_0(C)$ denote the probability that an empty box is selected. Since it is clear that any two ordered events in the same equivalence class have the same probability of occurring, we have, for $n_0 > 0$,

$$p(C, \alpha, +) = \frac{n!}{n_0! \cdots n_k!} p_1^{n_1} \cdots p_k^{n_k} (1 - p_1 - \cdots - p_k)^{n_0}, \tag{5}$$

whereas for $n_0 = 0$, and assuming C has no empty boxes,

$$p(C, \alpha, +) = \frac{n!}{n_1! \cdots n_k!} p_1^{n_1} \cdots p_{k-1}^{n_{k-1}} (1 - p_1 - \cdots - p_{k-1})^{n_k}. \tag{6}$$

In view of these formulas, the following lemma can be used to solve the above problem for certain events where, in contrast to the single draw problem, at least one ball of each color is to be drawn at some performance of the experiment.

LEMMA 1. *Let s_q be the closed and bounded set in R^q consisting of those points* $x = (x_1, \cdots, x_q)$ *such that* $0 \leqq x_i \leqq 1$, $i = 1, \cdots, q$, *and* $\sum_{i=1}^{q} x_i \leqq 1$. *Let* $f: s_q \to R$ *be defined by* $f(x_1, \cdots, x_q) = x_1^{r_1} x_2^{r_2} \cdots x_q^{r_q} (1 - x_1 - \cdots - x_q)^{r_{q+1}}$, *where* $r_i > 0$, $i = 1, \cdots, q + 1$. *Then the maximum value of f on s_q occurs* (*uniquely*) *at the point*

$$x = \left(\sum_{i=1}^{q+1} r_i \right)^{-1} (r_1, \cdots, r_q).$$

Proof. Now f is continuous and nonnegative on the compact set s_q, with $f \equiv 0$ on the boundary of s_q. Hence, f has an absolute maximum at some point in the interior of s_q, and this point must therefore be a *critical point* of the smooth function f. Setting the partial derivatives of f equal to zero, we see that a critical point in the interior of s_q must satisfy the system of q linear equations

$$r_i(x_1 + x_2 + \cdots + x_q) + r_{q+1} x_i = r_i, \qquad i = 1, 2, \cdots, q. \tag{7}$$

The coefficient matrix of the system (7) can be shown to have determinant equal to $(r_{q+1})^{r_q - 1}(r_1 + r_2 + \cdots + r_{q+1}) \neq 0$, and therefore there is a *unique* solution to (7), which is easily verified to be $x = (\sum_{i=1}^{q+1} r_i)^{-1}(r_1, \cdots, r_q)$. This completes the proof of the lemma.

Given $\alpha = (n_0, n_1, \cdots, n_k)$, let g.c.d. (α) = g.c.d. $\{n_0, n_1, \cdots, n_k\}$, where by convention, every integer divides 0. The following theorem gives a fairly general solution to the multiple repetition with replacement problem.

THEOREM 1. *Given* $\alpha = (n_0, n_1, \cdots, n_k)$, *suppose*

$$0 < \tilde{n}_i = \frac{n_i}{\text{g.c.d.}(\alpha)} \leqq m_i, \qquad i = 1, 2, \cdots, k.$$

If $n_0 = 0$, *a maximal configuration* $C^*(\alpha, +)$ *is obtained by taking* $\tilde{n}_1 + \tilde{n}_2 + \cdots + \tilde{n}_k$ *boxes, placing at least one ball of color* c_i *into* $\tilde{n}_i$ *of the boxes,* $i = 1, \cdots, k$, *subject to the restriction that no box contains balls of different colors. If* $n_0 > 0$, *a maximal configuration is obtained as above with the addition of* $\tilde{n}_0 = n_0/\text{g.c.d.}(\alpha)$ *empty boxes.*

Proof. If C^* is as described in the theorem, then

$$p_i = p_i(C^*, c_i) = \frac{n_i}{n}, \qquad (i = 1, \cdots, k), \tag{8}$$

and

$$p_0 = p_0(C^*) = \frac{n_0}{n}, \tag{9}$$

where $n = n_0 + n_1 + \cdots + n_k$. Since it is clear from Lemma 1 and formulas (5), (6) that any configuration achieving the "ideal" probabilities given in (8) and (9) is automatically a maximal configuration for $(\alpha, +)$, the proof of Theorem 1 is complete.

REMARKS. 1. Given $\alpha = (n_0, n_1, \cdots, n_k)$, an obvious necessary condition that a configuration exist which achieves the ideal probabilities (8) and (9) is that $n_i > 0$ for $i = 1, \cdots, k$. Events which do *not* satisfy these conditions are, in a sense, a generalization of the single draw problem. However, except for the case where $n_0 = 0$ and only one n_i is positive, $i = 1, \cdots, k$, (in which case a maximal configuration is obtained as in Section 2), the search for maximal configurations for these events appears to be guided primarily by the methods of Section 3. In particular, it appears that a maximal configuration C^* for $\alpha = (n_0, n_1, \cdots, n_k)$ must have a probability $(k+1)$-tuple $(p_0, p_1, \cdots, p_k)$ which does not differ by very much from the closest (using the usual distance in euclidean $(k+1)$-space) possible probability $(k+1)$-tuple to the ideal $(n_0/n, n_1/n, \cdots, n_k/n)$. That it *can* differ is illustrated by the example $k = 3$, $n = 2$, $m_1 = 3$, $m_2 = 2$, $m_3 = 2$, $\alpha = (0, 1, 1, 0)$, as the reader can verify (see the introduction for further discussion of this example).

2. A second situation which often, (but not always, as the example $k = 2$, $n = 4$, $m_1 = 2$, $m_2 = 1$, $\alpha = (0, 3, 1)$ shows), precludes the existence of a configuration

having the ideal probabilities $(n_0/n, n_1/n, \cdots, n_k/n)$ occurs when $n_i/\text{g.c.d.}(\alpha) > m_i$ for some $i > 0$. Again, for such events, configurations whose probabilities are closest to the ideal probabilities are not necessarily maximal, as illustrated by the example $k = 2$, $n = 2400$, $m_1 = 3$, $m_2 = 1$, $\alpha = (0, 1901, 499)$. Because of examples similar to those cited above, it appears unlikely that a concise, yet reasonably general theorem could be stated covering the multiple draw problem for events subject to either of the conditions $n_i = 0$, or $n_i/\text{g.c.d.}(\alpha) > m_i$, for some $i > 0$.

3. The definitions and problems of this section have analogous statements in the case where we are not replacing the balls after each draw. We make the transition notationally to this new setting by replacing the plus sign by a minus sign, so that we have $(C, n, —)$-events, etc. We have not worked out a general enough solution in the sampling without replacement problem to warrant recording here. Perhaps it should be noted that it can be shown that any two *nondegenerate* ordered $(C, n, —)$-events in the same equivalence class have the same probability of occurring, (the corresponding result for *degenerate* ordered $(C, n, —)$-events is false, in general), which considerably simplifies the search for maximal configurations.

CONTAGIOUS PROPERTIES

JAMES CHEW, Michigan State University

There are many instances in Topology when a property possessed by a subset of a space is also possessed by the closure of this subset. When P is such a property and a topological space has a dense subset having property P, then the entire space also enjoys property P. It seems appropriate to make the following definition:

DEFINITION 1. *A property of topological spaces is contagious if whenever a dense subspace of a space has the property, then it is true that the entire space also has the property.*

The purpose of this paper is to investigate which of the familiar properties encountered in Topology are contagious. Our first theorem is surely well known.

THEOREM 1. *Connectedness is contagious.*

If we restrict our spaces to Hausdorff spaces, of course it is well known that compactness is contagious. In fact we can get by with a milder separation axiom.

DEFINITION 2. *A space is $T_{1\frac{2}{3}}$ (or T_B) if given two disjoint compact sets A and B, there exist open sets U and V containing A and B respectively, such that $A \cap V = \varnothing$ and $B \cap U = \varnothing$.*

The symbol T_B was introduced by C. E. Aull [4], while the $T_{1\frac{2}{3}}$ terminology having the ideal probabilities $(n_0/n, n_1/n, \cdots, n_k/n)$ occurs when $n_i/\text{g.c.d.}(\alpha) > m_i$ for some $i > 0$. Again, for such events, configurations whose probabilities are closest to the ideal probabilities are not necessarily maximal, as illustrated by the example $k = 2$, $n = 2400$, $m_1 = 3$, $m_2 = 1$, $\alpha = (0, 1901, 499)$. Because of examples similar to those cited above, it appears unlikely that a concise, yet reasonably general theorem could be stated covering the multiple draw problem for events subject to either of the conditions $n_i = 0$, or $n_i/\text{g.c.d.}(\alpha) > m_i$, for some $i > 0$.

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The symbol T_B was introduced by C. E. Aull [4], while the $T_{1\frac{2}{3}}$ terminology appears in [3], p. 299. It turns out that T_B spaces are characterized by the fact that, in these spaces, compact sets are closed.

THEOREM 2. *A space is* $T_{1\frac{2}{3}}$ *(or* T_B*) iff its compact subsets are closed.*

Actually, Theorem 2 is taken as the *definition* of $T_{1\frac{2}{3}}$ spaces in [3]. It should now be clear that $T_2 \Rightarrow T_{1\frac{2}{3}} \Rightarrow T_1$ and that the following theorem holds:

THEOREM 3. *In the class of* T_B*-spaces, compactness is contagious.*

As an example of a T_B space that is not T_2, we mention the following. Let X = the reals and let $\mathscr{T} = \{\varnothing\} \cup \{U : (X - U)\,\text{countable}\}$. Then $(X, \mathscr{T})$ is T_B since the only compact sets are finite and $(X, \mathscr{T})$ is T_1. Clearly $(X, \mathscr{T})$ is not T_2 ([1], Example 20, p. 50).

Question. Can the T_B assumption be dropped from Theorem 3? (At least, can it be replaced by T_1?)

Perhaps it is too much to expect a property to be contagious without making some restriction on the type of spaces under consideration. Our *convention* is as follows. When we say property P is contagious without qualification we mean it is contagious in the class of all topological spaces; i.e., no restriction need be made on the spaces. If a restriction to a certain class of spaces is needed, we shall so specify. Compare, for example, the statements of Theorem 1 and Theorem 3.

Before stating the next theorem, we now recall the following:

DEFINITION 3. *A space* X *is pseudo-compact if each continuous real-valued function on* X *is bounded.*

THEOREM 4. *Pseudo-compactness is contagious.*

Proof. Suppose D is a dense pseudo-compact subspace of a topological space X. Let $f: X \to R$ be a continuous function from X to the reals. Then f_D, the restriction of f to D, is bounded. That is, there exists a natural number N such that $f_D(D) \subset (-N, N)$. Claim: $f(X) \subset (-N, N)$. Suppose for some $a \in X, f(a) \notin (-N, N)$. Let V be an open interval containing $f(a)$ such that $V \cap (-N, N) = \varnothing$. Let U be an open set containing a such that $f(U) \subset V$. Since D is dense in X, U must meet D. Let $d \in D \cap U$. Then $f(d) \in (-N, N)$ because $d \in D$, while $f(d) \in V$ because $d \in U$, a contradiction. ▮

Question. Is countable compactness contagious?

THEOREM 5. *Neither local connectedness nor local compactness is contagious.*

Proof. Let $D = \{(x, \sin(1/x)) : 0 < x \leqq 1\}$ and let $X = \{(0,0)\} \cup D$ be given the topology it inherits by virtue of its being a subset of the plane with the usual (euclidean) topology ([1], Example 116, pp. 137–8). ▮

It is well known that connectedness and compactness are continuous invariants. The next theorem is not quite as well known.

THEOREM 6. *Pseudo-compactness is a continuous invariant.*

Proof. Let $f: X \to Y$ be a continuous function from the pseudo-compact space X onto the space Y. Suppose $g: Y \to R$ is a real-valued function on Y. Then $gf: X \to R$ is a continuous real-valued function on X, and it follows from the pseudo-compactness of X that gf is bounded. Say $|(gf)(x)| \leqq M$ for each $x \in X$. Since f is onto, given $y \in Y$, there exists $x \in X$ such that $y = f(x)$. Hence given $y \in Y$, $|g(y)| = |g(f(x))| \leqq M$ and so g is bounded. ▮

The continuous image of a locally connected (locally compact) space may fail to be locally connected (locally compact), ([**1**], p. 138). Are all contagious properties continuous invariants? Here is a less sweeping question.

Question. Let X and Y be Hausdorff spaces and let $f: X \to Y$ be a continuous function from X onto Y. For which contagious properties P is it true that if X has property P then Y has property P also?

It would be nice to know an answer even under additional hypotheses on X and Y or f (for example, if f is closed or open or even a homeomorphism).

THEOREM 7. *T_i $(i = 1, 1\frac{2}{3}, 2, 3, 4)$ is not contagious.*

Proof. Let $N = \{1, 2, 3, \cdots\}$ and let $X = \{-1, 0\} \cup N$. Let $\mathscr{T}$ be defined as follows. Any subset of N belongs to $\mathscr{T}$. A set containing -1 or 0 belongs to $\mathscr{T}$ iff it contains all but a finite number of elements of N ([**1**], Example 27, p. 55). Since points are closed, $(X, \mathscr{T})$ is T_1. The set $C = \{0, 2, 4, \cdots\}$ is a compact subset of $(X, \mathscr{T})$. Since $X - C = \{-1, 3, 5, 7, \cdots\} \notin \mathscr{T}$, we see that C is not closed. This means $(X, \mathscr{T})$ is not T_B. Clearly N is a dense subset of $(X, \mathscr{T})$ and if $\mathscr{T}_N$ is the relative topology on N, then $(N, \mathscr{T}_N)$ is a discrete space. Hence $(N, \mathscr{T}_N)$ is T_i for $i = 0, 1, 1\frac{2}{3}, 2, 3, 4$. ▮

We cannot refrain from stating the corollary: *T_B is not contagious.*

Question. In the class of metric spaces, is completeness contagious?

The property of being Lindelöf is not contagious in general. However we show in [**2**] that the property of being Lindelöf is contagious in the class of paracompact spaces.

THEOREM 8. *In the class of paracompact spaces (no separation axiom needed), the property of being Lindelöf is contagious.*

Actually Theorem 8 is a corollary to the result in [**2**].

We wish to close this paper by mentioning two examples and stating a problem suggested by them. This last problem was the motivating force behind the idea of contagious properties. Let X = the reals and let $B = \{[a, b): a, b \in X,\ a < b\}$ be the basis for the topology $\mathscr{T}$ on X. We shall call $(X, \mathscr{T})$ the *Sorgenfrey line L* ([**1**], Example 51, p. 75). It is easy to see that L is a regular Hausdorff space as follows. If U is an open set containing an element a, then there exists $b > a$ such that $[a, b) \subset U$. But $[a, b)$ is also a closed set, so that $a \in [a, b) = \overline{[a, b)} \subset U$; this shows L is regular. That L is Hausdorff is clear. Since the set of rationals is dense in L, it follows that L is separable. Since each of the properties, regularity, Hausdorffness, separability is countably productive, it follows that the *Sorgenfrey plane* $S = L \times L$ ([1], Example 84, p. 103) is also a regular Hausdorff separable space.

Now separable spaces obviously have dense Lindelöf subspaces. Actually, separability is a *stronger* property than the property of having a dense Lindelöf subspace. It is known that L is Lindelöf while S is not. Thus the property of being Lindelöf is *not* contagious even in the presence of rather strong separation conditions (both L and S are completely regular Hausdorff spaces, for example). The point is it is easy to spot a dense Lindelöf (in fact, countable) subspace in both L and S, while the proof that L is Lindelöf is difficult since it usually requires a category argument. It is in connection with determining whether a space is Lindelöf that the idea of contagious properties is useful.

Question. When is a separable regular Hausdorff space Lindelöf?

Note that the above question asks when is the property of being Lindelöf contagious in the presence of a great deal of assumptions on the class of spaces under consideration. Of course, we know that the property of being Lindelöf is *not* contagious in the class of completely regular Hausdorff, separable spaces. The Sorgenfrey plane S is a counterexample.

An answer to the last question is: When the space is metacompact ([1], p. 24), in which case only the separability of the space is used. The trouble is that determining whether the Sorgenfrey line L is metacompact seems to be as difficult as determining whether it is Lindelöf. A "good" answer would be one which would enable us to deduce that L is Lindelöf while S is not from easily deducible properties of L and S.

References

1. L. A. Steen and J. A. Seebach, Counterexamples in Topology, Holt, Rinehart and Winston, New York, 1970.
2. J. Chew, Note on "Paracompactness in small products", Canad. Math. Bull., 163 (1973).
3. H. Cullen, Introduction to General Topology, D. C. Heath, Boston, 1967.
4. C. E. Aull, Separation of bicompact sets, Math. Ann., 158 (1965) 197–202.

ON THE STEINER-LEHMUS THEOREM

MORDECHAI LEWIN, Israel Institute of Technology

In the year 1840 Professor Lehmus from Berlin asked the famous Swiss geometer Jacob Steiner for a proof of the proposition stating that if two angle-bisectors in a triangle are equal, the triangle is isosceles. Steiner soon found a proof but did not publish it until 1844 [5]. In 1850 Lehmus found a proof of his own. However it was the French mathematician Rougevain who was the first to publish a proof in 1842.

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The theorem has been tackled ever since by many people, among them renowned mathematicians. Around 1850 the problem entered England with the additional requirement that the proof be direct as all the proofs up to then had been by *reductio ad absurdum*. Many proofs followed, among them a considerable number of incorrect proofs. In 1882 Sylvester [6] intended to prove the impossibility of a direct proof. Although Sylvester's proof of this was not really a proof in the strict sense, the conclusion nevertheless seemed accepted. Again after some period there began to appear so called direct proofs of the Steiner-Lehmus Theorem. The seemingly never-ending stream culminated in a paper by Henderson [3] whose avowed aim it was "to write an essay on the internal bisector problem to end all essays on the internal bisector problem." To strengthen his point he supplied as many as ten different proofs, seven "direct" and three indirect ones.

In 1943 there appeared a centenary account by McBride [4] mentioning about sixty different proofs and defending the attitude of Sylvester as regards direct proofs.

In 1961 there appeared a simple (indirect) proof in Coxeter's book *Introduction to Geometry*. In the same year there appeared an enthusiastic review of Coxeter's book by Martin Gardner. As a feature Gardner referred to the simple proof of the Steiner-Lehmus Theorem. As a result of the review, the reviewer came under fire of hundreds of proofs among which after careful deliberation one single proof (by Gilbert and McDonnell) was chosen and published in the AMERICAN MATHEMATICAL MONTHLY [2]. The circle closed when it turned out after publication that the proof that had been so carefully selected was essentially identical with the original proof of Lehmus in 1850.

In March 1970 issue of this MAGAZINE there appeared a proof of the Steiner-Lehmus Theorem by Malesevic [7] claiming to be direct. Malesevic's proof, although not short, is certainly interesting. There are however various reasons for challenging the directness of the proof. The present note proposes to analyze the crucial points of the proof.

1. "From the fact that aιea $\triangle ABE$ = area $\triangle ECN$. . ." What is this fact based upon? Apparently since $AB = CN$ and E is equidistant from AB and BC, this last assertion is true, but why? Because of the fourth theorem of congruence which asserts that two triangles are congruent if they are equal in two sides and the angle opposite the greater of the two. However there does not seem to exist a direct proof of this fourth theorem. The same goes for ". . .area $\triangle ABD$ = area $\triangle DCM$."

2. ". . .which implies that $\measuredangle EON = \measuredangle DOM$." The implication uses the same crucial fourth theorem of congruence.

3. "$\measuredangle CAB = \measuredangle CBA$." Again true but what is it based upon? The outer angle in a triangle equals the sum of the two inner angles not adjacent to the outer angle! And this is based on Theorem I 29 of Euclid's Elements: *Corresponding angles between parallels are equal.* And how does one prove this? There is no known direct proof of Euclid's theorem.

4. ". . .which was to be proved." What was to be proved? That the triangle ABC is isosceles and not equality of the two angles mentioned in point 3. The argument

of course is that equality of the angles implies equality of the corresponding sides. This is Theorem I 6 in Euclid. Unfortunately Euclid's theorem is proved indirectly.

It might help anyone who is interested in the problem to refer to the article of McBride [4] where any "direct" proof is rightly discredited if it depends on indirectly proved lemmas. Personally I am inclined to attribute considerable weight to the concluding remark of McBride:

"If it is held, as I hold, that Euc. I. 14, Euc. I. 29, Euc. I. 32, and the Theorem of Pythagoras have no direct proof, then the Bisector Theorem has not been proved directly, nor is it likely to be."

References

1. H. S. M. Coxeter, Introduction to Geometry, Wiley, New York, 1961.

2. G. Gilbert and D. McDonnell, The Steiner-Lehmus theorem, Amer. Math. Monthly, 70 (1963) 79–80.

3. A. Henderson, A classic problem in Euclidean geometry, J. Elisha Mitchell Soc., (1937) 246–281.

4. J. A. McBride, The equal internal bisectors theorem, Proc. Edinburgh Math. Soc. Edinburgh Math. Notes, 33 (1943) 1–13.

5. J. Steiner, The solution in Crelle's J. XXVIII (1844), (reproduced in Steiner's Gesammelte Werke II 323-b (1882)).

6. J. J. Sylvester, On a simple geometrical problem illustrating a conjectured principle in the theory of geometrical method, Phil. Magazine, 4 (1852) 366–369.

7. J. V. Malesevic, A direct proof of the Steiner-Lehmus theorem, this MAGAZINE, 43 (1970) 101–102.

ON THE WRONSKIAN FORMULA

MAURICE MACHOVER, St. John's University

The basic Wronskian formula for a system of n linear first order differential equations is [1]

$$W(t) = W(t_0) \exp \int_{t_0}^{t} \left(\sum_{k=1}^{n} a_{kk}(\xi) \right) d\xi, \tag{1}$$

where $W(t) = \det \Phi(t)$ is the Wronskian of any n by n solution matrix $\Phi(t)$ of

$$\Phi' = A(t)\Phi, \tag{2}$$

and $\sum_{k=1}^{n} a_{kk}(t) = \operatorname{tr} A(t)$ is the trace of $A(t)$. We discuss here some generalizations of this formula for certain special forms of Φ and A and present counterexamples to show that these generalizations are best possible. Also we present a different proof of the formula by passing from triangular matrices to general matrices via a continuity argument in the Jordan canonical form.

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THEOREM. *Let $\Phi(t)$ be a nonsingular solution matrix of* (2). *Then*

$$(\det \Phi)' = (\operatorname{tr} A)(\det \Phi). \tag{3}$$

Proof. We first consider the case when Φ is a triangular matrix J with diagonal elements $j_1(t), j_2(t), \cdots, j_n(t)$. From (2) $A = J'J^{-1}$. By the elementary facts for triangular matrices J', J^{-1}, and A have diagonal elements $j_1', j_2', \cdots, j_n'$; $1/j_1, 1/j_2, \cdots, 1/j_n$; and $j_1'/j_1, j_2'/j_2, \cdots, j_n'/j_n$, respectively.

Thus $\operatorname{tr} A = \sum_{k=1}^n j_k'/j_k$ and $\det J = \prod_{k=1}^n j_k$. Since

$$\left(\prod_{k=1}^n j_k\right)' = \left(\prod_{k=1}^n j_k\right)\left(\sum_{k=1}^n j_k'/j_k\right)$$

is a simple identity for derivatives, the result follows in this case.

For the general case we use the Jordan form of Φ. Let $P(t)$ be a nonsingular matrix such that $P^{-1}\Phi P = J$ is in Jordan form (and thus is triangular). Thus $\det \Phi = \det J$ and $A = \Phi'\Phi^{-1}$. Since $\Phi = PJP^{-1}$ it follows that $\Phi^{-1} = PJ^{-1}P^{-1}$ and

$$\begin{aligned}\Phi'\Phi^{-1} &= (P'JP^{-1} + PJ'P^{-1} - PJP^{-1}P'P^{-1})(PJ^{-1}P^{-1}) \\ &= P'P^{-1} + PJ'J^{-1}P^{-1} - PJP^{-1}P'J^{-1}P^{-1}.\end{aligned}$$

Accordingly, $\operatorname{tr} A = \operatorname{tr}(\Phi'\Phi^{-1}) = \operatorname{tr}(P'P^{-1}) + \operatorname{tr}(J'J^{-1}) - \operatorname{tr}(P^{-1}P') = \operatorname{tr}(J'J^{-1})$. From the triangular case we conclude that $(\det \Phi)' = (\det J)' = \operatorname{tr}(J'J^{-1})(\det J) = (\operatorname{tr} A)(\det \Phi)$, proving the theorem (and establishing (1) by an integration).

(The nondifferentiability of $P(t)$ in the case that $\Phi(t)$ has a multiple eigenvalue at say t_0 is no barrier. For any multiple root $\lambda(t_0)$ of $\det[\Phi(t_0) - \lambda I] = 0$ bifurcates into simple roots for t near t_0 [**2**, p. 122]. Thus (3) can be established at t_0 by a continuity argument, i.e., by letting $t \to t_0$.)

For certain special forms of Φ and A the Wronskian formula (1) can be generalized. Thus it is well known that if $A(t)$ commutes with $\int_{t_0}^t A(\xi)d\xi$ then

$$\Phi(t) = \exp\left(\int_{t_0}^t A(\xi)d\xi\right)\Phi(t_0). \tag{4}$$

This commutativity condition is by no means necessary as is illustrated by the choice (for $n = 2$)

$$A(t) = \begin{bmatrix} 0 & t \\ 0 & 1 \end{bmatrix}, \quad \Phi(t) = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \quad t_0 = 0. \tag{5}$$

The basic formula (1) expresses the fact that the determinants of the matrices in (4) are equal in general.

Even for triangular matrices (4) does not always hold. This is illustrated by the choice

$$A(t) = \begin{bmatrix} 0 & t \\ 0 & 1 \end{bmatrix}, \quad \Phi(t) = \begin{bmatrix} 0 & te^t - e^t \\ 0 & e^t \end{bmatrix}, \quad t_0 = 0 \tag{6}$$

Here

$$\exp\left(\int_{t_0}^{t} A(\xi)d\xi\right) = \begin{bmatrix} 1 & \frac{1}{2}(te^t - t) \\ 0 & e^t \end{bmatrix}. \tag{7}$$

Nevertheless, since the usual operations on these matrices are preserved along the diagonal it can be expected that at least the characteristic polynomials of the matrices in (4) are equal. That this is so is shown in the following theorem:

THEOREM. *Let $\Phi(t)$ be a nonsingular triangular solution matrix of* (2). *Then*

$$\det[\Phi(t) - \lambda I] = \det\left[\exp\left(\int_{t_0}^{t} A(\xi)d\xi\right)\Phi(t_0) - \lambda I\right]. \tag{8}$$

Proof. Let Φ have diagonal elements $j_1, j_2, \cdots, j_n$. Then $A = \Phi'\Phi^{-1}$ has diagonal elements $j_1'/j_1,\ j_2'/j_2, \cdots, j_n'/j_n$. It follows that $\int_{t_0}^{t} A(\xi)d\xi$ has diagonal elements $\ln(j_1(t)/j_1(t_0)),\ \ln(j_2(t)/j_2(t_0)), \cdots, \ln(j_n(t)/j_n(t_0))$, so that $\exp(\int_{t_0}^{t} A(\xi)d\xi)\Phi(t_0)$ is also triangular with diagonal elements $j_1, j_2, \cdots, j_n$. Both sides of (8) are therefore equal to $\prod_{k=1}^{n}(j_k - \lambda)$.

The fact that triangularity of Φ is needed in the above theorem is illustrated by the choice

$$A(t) = \begin{bmatrix} -1 & 2e^t \\ 0 & 0 \end{bmatrix},\quad \Phi(t) = \begin{bmatrix} e^t & e^{-t} \\ 1 & 0 \end{bmatrix},\quad t_0 = 0. \tag{9}$$

Here

$$\exp\left(\int_{t_0}^{t} A(\xi)d\xi\right) = \begin{bmatrix} e^{-t} & (-1/t)(2e^t - 2)(e^{-t} - 1) \\ 0 & 1 \end{bmatrix}, \tag{10}$$

so that not even the traces of the matrices in (4) are equal (although their determinants are by (1)).

References

1. F. John, Ordinary Differential Equations, Lecture Notes: New York University, 1965.
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ON THE SMALLEST PRIME GREATER THAN A GIVEN POSITIVE INTEGER

STEVEN KAHAN, Queens College, Flushing, New York

At this writing, the existence of a prime greater than or equal to a positive integer $m \neq 8$ and less than or equal to the number $m + \sqrt{m}$ is an open question in the theory of numbers. If we accept the truth of this conjecture, we can then prove the following:

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Proof. Let Φ have diagonal elements $j_1, j_2, \cdots, j_n$. Then $A = \Phi'\Phi^{-1}$ has diagonal elements $j_1'/j_1, j_2'/j_2, \cdots, j_n'/j_n$. It follows that $\int_{t_0}^{t} A(\xi)d\xi$ has diagonal elements $\ln(j_1(t)/j_1(t_0)), \ln(j_2(t)/j_2(t_0)), \cdots, \ln(j_n(t)/j_n(t_0))$, so that $\exp(\int_{t_0}^{t} A(\xi)d\xi)\Phi(t_0)$ is also triangular with diagonal elements $j_1, j_2, \cdots, j_n$. Both sides of (8) are therefore equal to $\prod_{k=1}^{n}(j_k - \lambda)$.

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THEOREM. *Let* $n \geqq 2$ *be a positive integer and let* $N_n = \{1, 2, \cdots, n\}$. *Let* $A = \bigcup_{i=2}^{n} \{i[(n+i)/i] - n\}$ *and let* $B = N_n - A$. *Define* $p(n) = n + \min B$. *Then* $p(n)$ *is the smallest prime* $> n$. *In particular, if* $n = p_j$, *the* jth *prime, then* $p(n) = p_{j+1}$.

Proof. The theorem may be verified directly for $n = 7$ without appeal to the conjecture, so we will assume that $n \neq 7$.

Let p^* be the smallest prime $> n$. Then $p^* = n + s$, where $1 \leqq s < n$, since Bertrand's Postulate guarantees the existence of a prime between n and $2n$. (A proof of Bertrand's Postulate may be found in *An Introduction to the Theory of Numbers* by R. G. Archibald, as well as in other standard texts.) To show $p(n) = p^*$, it suffices to show that $s = \min B$.

Suppose that for some i, $2 \leqq i \leqq n$, it is true that $s = i[(n+i)/i] - n$. Then $p^* = n + s = i[(n+i)/i]$, contradicting the fact that p^* is a prime. Hence, $s \notin A$. Since $s \in N_n$, it follows that $s \in B$.

If $s = 1$, then clearly $s = \min B$. So without loss of generality, assume $s \geqq 2$. Then it remains to show that for any t, $1 \leqq t < s$, $t \notin B$, i.e., for any t, $1 \leqq t < s$, $t \in A$. In other words, we must show that given any t, $1 \leqq t < s$, there exists an i, $2 \leqq i \leqq s$, such that $t = i[(n+i)i/] - n$, i.e., such that $n + t = i[(n+i)/i]$. Note that since $n < n + t < n + s$, it follows that $n + t$ is composite.

Claim 1. If f is a nontrivial (i.e., distinct from 1 and $n + t$) factor of $n + t$, then $f \leqq n$.

Proof of Claim 1. Write $n + t = fg$, and suppose $f > n$. Then $fg > ng$. Since f is a nontrivial factor of $n + t$, so is g. Hence, $g \geqq 2$, from which it follows that $ng \geqq 2n$. By transitivity, $fg > 2n$, i.e., $n + t > 2n$. Therefore $t > n$. But $t < s$. Again by transitivity, $n < s$, a contradiction. We can thus conclude that $f \leqq n$.

Claim 2. If f is the largest nontrivial factor of $n + t$, then $n + t = f[(n+f)/f]$.

Proof of Claim 2. Consider the identity $(n+t)/f + (f-t)/f = (n+f)/f$. It follows immediately that

$$\left[\frac{n+t}{f} + \frac{f-t}{f}\right] = \left[\frac{n+f}{f}\right].$$

Since the hypothesis gives f to be a factor of $n + t$, $(n+t)/f$ is an integer, so

$$\left[\frac{n+t}{f} + \frac{f-t}{f}\right] = \frac{n+t}{f} + \left[\frac{f-t}{f}\right].$$

Hence,

$$\frac{n+t}{f} + \left[\frac{f-t}{f}\right] = \left[\frac{n+f}{f}\right], \text{ i.e., } n + t = f\left[\frac{n+f}{f}\right] - f\left[\frac{f-t}{f}\right].$$

The claim will follow if we can show that $[(f-t)/f] = 0$.

Since f is the largest nontrivial factor of $n + t$, it follows that $f \geqq \sqrt{n+t} \geqq \sqrt{n+1}$.

Also, since $t < s$, the elements of the set $T = \{n+1, \cdots, n+t\}$ are all composites. Now suppose that $t \geqq 1 + \sqrt{n+1}$. Then the set of integers greater than or equal to $n+1$ and less than or equal to $n+1+\sqrt{n+1}$ is a subset of T. Since $n \neq 7$, the conjecture implies the existence of a prime in this subset, hence in T; this contradicts the fact that T contains only composites. Therefore, $t < 1+\sqrt{n+1}$, so $t - 1 < \sqrt{n+1}$. Then by transitivity, $t - 1 < f$, which implies that $t \leqq f$. So $0 \leqq f - t$. Clearly, $f - t < f$. Hence, $0 \leqq f - t < f$. Dividing by f, $0 \leqq (f-t)/f < 1$, which implies $[(f-t)/f] = 0$. This completes the proof of Claim 2.

Since $2 \leqq f \leqq n$, choose $i = f$, the largest nontrivial factor of $n + t$. Then by the previous remarks, it follows that $p(n) = p^*$, proving the theorem.

THE BEAUTY, THE BEAST, AND THE POND

W. SCHUURMAN and J. LODDER, FOM-Institute for Plasmaphysics, Rijnhuizen, Jutphaas, Netherlands

An old puzzle (see Ref. **1**) deals with a female beauty B situated in a row boat at the center of a circular pond (radius R), and a male aggressor A running along the circumference of the pond at maximum speed v. B rows to the shore with speed αv where α is a given fixed number. The problem is: can she prevent A from grabbing her (assuming she is the better runner) and what should be the poor girl's strategy? (see Figure 1).

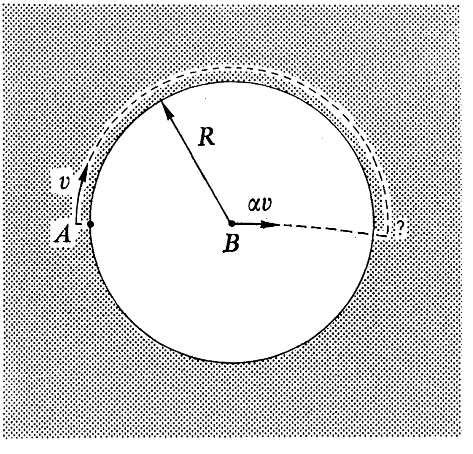

FIG. 1

However, the problem can be stated more generally: for which values of α can B escape, and for each such α what is the corresponding strategy? This generalized puzzle, much more difficult to solve, is briefly discussed here.

Also, since $t < s$, the elements of the set $T = \{n+1, \cdots, n+t\}$ are all composites. Now suppose that $t \geqq 1 + \sqrt{n+1}$. Then the set of integers greater than or equal to $n+1$ and less than or equal to $n + 1 + \sqrt{n+1}$ is a subset of T. Since $n \neq 7$, the conjecture implies the existence of a prime in this subset, hence in T; this contradicts the fact that T contains only composites. Therefore, $t < 1 + \sqrt{n+1}$, so $t - 1 < \sqrt{n+1}$. Then by transitivity, $t - 1 < f$, which implies that $t \leqq f$. So $0 \leqq f - t$. Clearly, $f - t < f$. Hence, $0 \leqq f - t < f$. Dividing by f, $0 \leqq (f-t)/f < 1$, which implies $[(f-t)/f] = 0$. This completes the proof of Claim 2.

Since $2 \leqq f \leqq n$, choose $i = f$, the largest nontrivial factor of $n + t$. Then by the previous remarks, it follows that $p(n) = p^*$, proving the theorem.

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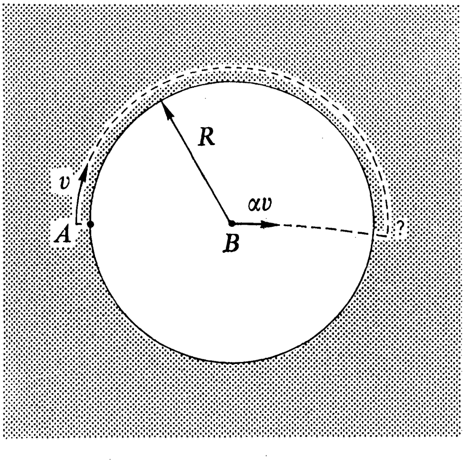

FIG. 1

However, the problem can be stated more generally: for which values of α can B escape, and for each such α what is the corresponding strategy? This generalized puzzle, much more difficult to solve, is briefly discussed here.

The solution evolves through several levels of sophistication. If $\alpha > \pi^{-1}$ a panicky and therefore clumsy method of escape suffices: rowing in a straight line towards A's original antipoint. For values of α between π^{-1} and $(\pi + 1)^{-1}$ the beauty will have to use her brains (if endowed with them). She has to introduce the concept of the limiting circle. By choosing the θ-component of her velocity, v_θ, equal to $(r/R)v$ she can remain diametrically opposite to A until she reaches the circle $r = \alpha R$. This circle constitutes a limit, since obviously $v_\theta \leqq \alpha v$. The time t needed to reach this circle is finite:

$$t = \int_0^{\alpha R} \left(\alpha^2 v^2 - \frac{r^2}{R^2} v^2\right)^{-1/2} dr = \frac{\pi R}{2v}.$$

She can then complete the escape by crossing the rest of the pond radially.

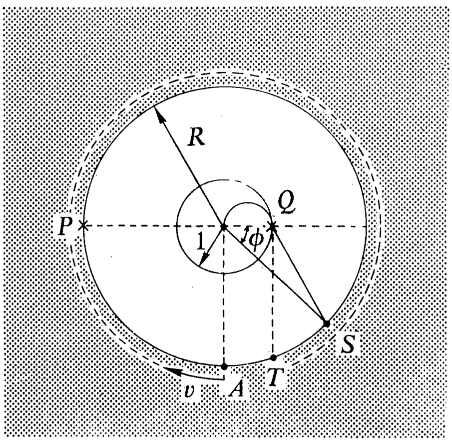

FIG. 2

At first sight it seems that no method of escape exists if $\alpha \leqq (\pi + 1)^{-1}$. Further consideration, however, shows that the trick can still be done. First, B rows to the diametrical position Q on the limit circle (see Figure 2). Then she performs an infinitesimal radial feint in order to induce A to choose a definite sense of rotation along the shore, starting at P. Let him choose the clockwise direction. From that moment on, A's best policy is to continue running clockwise if B goes to the shore along a straight line (always better than a curved one!) *not crossing the limit circle.* If A would return, a new diametrical mutual position, advantageous to B would be established. The final problem is to optimize the direction of B's straight track to the shore.

Since

$$\frac{r\,(\text{limit circle})}{R} = \frac{\text{speed } (B)}{\text{speed } (A)} = \alpha,$$

we may artificially call r (limit circle) $= 1$ and ask for the maximum value of R allowing B to escape. B rows from point Q on the limit circle along the straight line QS and reaches S an infinitesimal interval of time earlier than A (see Figure 2).

A's path length $PS = R\ (\pi + \phi)$ requires the time $R(\pi + \phi)/v$ and B's path length $QS = (R^2 + 1 - 2R\cos\phi)^{1/2}$ requires the time $(R/v)\ (R^2 + 1 - 2R\cos\phi)^{1/2}$. Equating the two time intervals we obtain $\pi + \phi = (R^2 + 1 - 2R\cos\phi)^{1/2}$ or $R = \cos\phi + \{\cos^2\phi + (\pi + \phi)^2 - 1\}^{1/2}$. This R is a monotonic nondecreasing function of ϕ. Because of the prohibition against crossing the unit circle, it follows that the tangent QT to the shore is the best course to be taken. The tangent solution leads to the equations $\tan\phi = \pi + \phi$ and $\cos\phi = R^{-1}$. Numerical solution gives $R_{\max} = 4.603338849\cdots$. Thus, the girl B, in her predicament, will be able to escape provided the speed of the villain is less than $4.603338849\cdots$ times her speed of rowing.

This work was performed as part of the recreational program of the association agreement of Euratom and the "Stichting voor Fundamenteel Onderzoek der Materie" (FOM) with financial support from the "Nederlandse Organisatie voor Zuiver-Wetenschappelijk Onderzoek" (ZWO) and Euratom.

Reference

1. Scientific American, 213 (1965) 116.

SECOND ORDER PERIMETER-MAGIC AND PERIMETER-ANTIMAGIC CUBES

CHARLES W. TRIGG, San Diego, California

The first eight positive digits can be distributed on the vertices of a cube in 1680 ways, not counting rotations. That is, on the vertices connected to the vertex containing the digit 1, any set of three digits can be placed in only two distinct cyclic permutations. Hence the total of distinct distributions is $[C(7,3)](2)(4!)$ or 1680. Since only two integers occur on an edge, the distribution is a *second order* distribution.

If each vertex digit is subtracted from 9, a new distribution (cube) is obtained which is the *complement* of the original distribution (cube). If a cube is identical with its complement or the mirror image of its complement, the cube is *self-complementary*.

Perimeter-magic cubes. When the sums of the vertex digits on the perimeters of the six faces are equal, the cube will be called *V-type perimeter-magic* or *second order perimeter-magic* (as distinguished from edge-magic and face-magic).

The sum of the first eight positive digits is 36. Since opposite faces have no digits in common, the perimeter-magic sum is 18. So, the complement of a perimeter-magic cube is perimeter-magic also.

The eight partitions of 18 into four distinct positive digits < 9 occur in four conjugate pairs, namely:

A's path length $PS = R(\pi + \phi)$ requires the time $R(\pi + \phi)/v$ and B's path length $QS = (R^2 + 1 - 2R\cos\phi)^{1/2}$ requires the time $(R/v)\ (R^2 + 1 - 2R\cos\phi)^{1/2}$. Equating the two time intervals we obtain $\pi + \phi = (R^2 + 1 - 2R\cos\phi)^{1/2}$ or $R = \cos\phi + \{\cos^2\phi + (\pi + \phi)^2 - 1\}^{1/2}$. This R is a monotonic nondecreasing function of ϕ. Because of the prohibition against crossing the unit circle, it follows that the tangent QT to the shore is the best course to be taken. The tangent solution leads to the equations $\tan\phi = \pi + \phi$ and $\cos\phi = R^{-1}$. Numerical solution gives $R_{max} = 4.603338849\cdots$. Thus, the girl B, in her predicament, will be able to escape provided the speed of the villain is less than $4.603338849\cdots$ times her speed of rowing.

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CHARLES W. TRIGG, San Diego, California

The first eight positive digits can be distributed on the vertices of a cube in 1680 ways, not counting rotations. That is, on the vertices connected to the vertex containing the digit 1, any set of three digits can be placed in only two distinct cyclic permutations. Hence the total of distinct distributions is $[C(7,3)](2)(4!)$ or 1680. Since only two integers occur on an edge, the distribution is a *second order* distribution.

If each vertex digit is subtracted from 9, a new distribution (cube) is obtained which is the *complement* of the original distribution (cube). If a cube is identical with its complement or the mirror image of its complement, the cube is *self-complementary*.

Perimeter-magic cubes. When the sums of the vertex digits on the perimeters of the six faces are equal, the cube will be called *V-type perimeter-magic* or *second order perimeter-magic* (as distinguished from edge-magic and face-magic).

The sum of the first eight positive digits is 36. Since opposite faces have no digits in common, the perimeter-magic sum is 18. So, the complement of a perimeter-magic cube is perimeter-magic also.

The eight partitions of 18 into four distinct positive digits < 9 occur in four conjugate pairs, namely:

1 2 7 8	3 4 5 6
1 3 6 8	2 4 5 7
1 4 5 8	2 3 6 7
1 4 6 7	2 3 5 8

Only three of these conjugate pairs can appear simultaneously on the faces of a cube. Since one of the partitions of each of the first three pairs contains 1 and 8 these pairs cannot all appear together. Consequently, 1 4 6 7 must appear on one face and 2 3 5 8 on the opposite face of any perimeter-magic cube.

There are six cyclic permutations of 1 4 6 7. With each permutation on a face, there is only one way that 2 3 5 8 can be distributed on the opposite face so that the connecting faces are occupied by other conjugate pairs. In every case, the complementary digits in the two partitions originally distributed are joined by common edges. The six perimeter-magic cubes are self-complementary and exist in mirror image pairs. One of each pair is shown in the Schlegel diagrams of Figure 1.

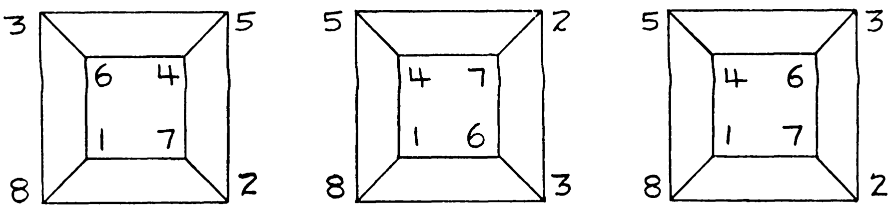

FIG. 1. Perimeter-magic cubes

Perimeter-antimagic cubes. When the sums of the digits on the perimeters of the six faces are distinct, the cube is *perimeter-antimagic*. 582 of the 1680 second order digited cubes fall into this category. On a few of them the sums are in arithmetic progression. Each vertex digit appears on three perimeters, so

$$3[8(1+8)/2] = 6(2a+5d)/2.$$

That is, $2a + 5d = 36$, so d is even.

If $d = 2$, then $a = 13$, and the other terms of the progression are 15, 17, 19, 21 and 23. This is the only possible progression, since if $d = 4$ then $a = 8$, but no perimeter can be less than 10.

Now 13 can be partitioned into four distinct positive digits < 9 in only three ways, as can 23. These partitions must lie on opposite faces. The conjugate pairs are

1 2 3 7	4 5 6 8
1 2 4 6	3 5 7 8
1 3 4 5	2 6 7 8

The cyclic permutations of the conjugate partitions are matched to find distributions for which the perimeters of the other four faces have the values of the intermediate terms of the progression. The 26 perimeter-antimagic cubes with sums in arithmetic progression occur as mirror image pairs. Of the 13, there are 6 complementary pairs (of which one member is given in Table 1) and 1 self-complementary cube, the last one in Table 1. In Table 1, the digits a, b, c, d proceed clockwise on the top face of the cube, and the digits directly below them on the cube are a', b', c', d', respectively.

TABLE 1

Basic Perimeter-Antimagic Cubes with Sums in Arithmetic Progression

a	b	c	d	a'	b'	c'	d'
1	2	3	7	4	8	6	5
1	2	3	7	8	6	4	5
1	3	2	7	5	6	8	4
1	3	7	2	6	5	4	8
1	3	7	2	8	5	6	4
1	4	2	6	5	7	8	3
1	4	2	6	7	3	8	5

RINGS WHOSE IDEALS FORM A CHAIN

E. T. HILL, Cornell College

It is well known [4, Corollary 7.31] that if a cyclic group of order p^n (p a prime) is generated by an element a, then every subgroup is generated by an element of the form a^{p^i} for some i. Since any group of order p^n has subgroups of order p^i for $0 \leqq i \leqq n$, the subgroups of the cyclic group form a chain and the chain is as long as possible consistent with the order of the group. Using the Sylow Theorems and the Burnside Basis Theorem [1], one readily shows the converse; i.e., if the subgroups of a finite group form a chain, then the group is a cyclic p-group. Given these results, it is natural to ask if there are rings such that the lattice of ideals is a chain of maximal length and, conversely, if the lattice of ideals has this property, what does it tell us about the ring?

One way to produce a ring whose ideals form a chain is to take an additive cyclic group of order p^n and define a zero multiplication. A more interesting example may be formed using group rings. A group ring (or algebra) is formed by using the members of the group as a basis for a vector space over some field; the group multiplication is then extended in the natural way to form a ring by insisting that the distributive law

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One way to produce a ring whose ideals form a chain is to take an additive cyclic group of order p^n and define a zero multiplication. A more interesting example may be formed using group rings. A group ring (or algebra) is formed by using the members of the group as a basis for a vector space over some field; the group multiplication is then extended in the natural way to form a ring by insisting that the distributive law hold. If one forms the group ring of a group of order p^n over Z_p (the integers modulo p where p is prime), then the radical, $\mathcal{N}$, of the ring is the set of nilpotent elements [3]. Letting $\mathcal{N}^w$ be the set of sums of products of w elements of $\mathcal{N}$, it is clear that $\mathcal{N}^w$ is an ideal and there is an integer L, called the exponent of $\mathcal{N}$, such that $\mathcal{N}^L \neq 0$ while $\mathcal{N}^{L+1} = 0$. In [2] it is shown that the only ideals in the group ring of a cyclic group of order p^n are the radical powers and the exponent of the radical is $L = p^n - 1$. Hence the only ideals are the $\mathcal{N}^w$ for $0 \leqq w \leqq p^n$ ($\mathcal{N}^0$ is the entire ring) and $\mathcal{N}^w / \mathcal{N}^{w+1}$ has p members so we have a nontrivial example of a ring whose lattice of ideals is a chain of maximal length.

A partial converse is given by the following:

THEOREM. *Let $\mathfrak{A}$ be an algebra over Z_p with unity and radical $\mathcal{N}$. If the exponent of $\mathcal{N}$ is $L = p^n - 1$ and $\mathcal{N}^i / \mathcal{N}^{i+1}$ has p elements for $0 \leqq i \leqq p^n - 1$, then $\mathfrak{A}$ is the group algebra of a cyclic group of order p^n over Z_p.*

Proof. Choose α in $\mathcal{N}$, α not in $\mathcal{N}^2$. The elements of $\mathcal{N}$ are then $0, \alpha, \cdots, (p-1)\alpha$ mod $\mathcal{N}^2$; hence, any β_i in $\mathcal{N}$ may be written $\beta_i = a_i \alpha + \gamma_i$ where a_i is in Z_p and γ_i is in $\mathcal{N}^2$. Since the members of $\mathcal{N}^{p^n-1}$ are linear combinations of products of the form

$$\prod_{i=1}^{p^n-1} \beta_i = \prod_{i=1}^{p^n-1} (a_i\alpha + \gamma_i) \equiv \left(\prod_{i=1}^{p^n-1} a_i\right) \alpha^{p^n-1} \bmod \mathcal{N}^{p^n}$$

and $\mathcal{N}^{p^n} = 0$, we must have $\alpha^{p^n-1} \neq 0$ because $\mathcal{N}^{p^n-1} \neq 0$. Therefore, α^i is in $\mathcal{N}^i$ and not in $\mathcal{N}^{i+1}$.

Suppose a linear combination of the α^i is given by

$$0 = a_0 + a_1\alpha + a_2\alpha^2 + \cdots + a_{p^n-1}\alpha^{p^n-1}.$$

Since this means that $a_0 = -\sum_{i=1}^{p^n-1} a_i\alpha^i$, it follows that a_0 is in $\mathcal{N} \cap Z_p$; i.e., $a_0 = 0$. If $a_i = 0$ for $i < k$, then $a_k\alpha^k = -\sum_{i=k+1}^{p^n-1} a_i\alpha^i$ so that either a_k is not a unit or α^k is in $\mathcal{N}^{k+1}$; that is, $a_k = 0$. Hence all the a_i are zero and $\{1, \alpha, \alpha^2, \cdots, \alpha^{p^n-1}\}$ is a linearly independent set. This implies $\{1, (1-\alpha), (1-\alpha)^2, \cdots, (1-\alpha)^{p^n-1}\}$ is linearly independent and, since it has p^n members, it is a basis for $\mathfrak{A}$. Notice that $(1-\alpha)^{p^n} = 1 - \alpha^{p^n}$ since all except the first and last coefficients in the binomial expansion are congruent to 0 mod p. Also, $\alpha^{p^n} = 0$ so that $(1-\alpha)^{p^n} = 1 - \alpha^{p^n} = 1$. Hence the set $\{1, (1-\alpha), \cdots, (1-\alpha)^{p^n-1}\}$ is a cyclic group which is a basis for $\mathfrak{A}$ and the theorem is proved.

References

1. M. Hall, The Theory of Groups, Macmillan, New York, 1959.

2. E. T. Hill, The annihilator of radical powers in the modular group ring of a p-group, Proc. Amer. Math. Soc., 25, 4 (1970) 811–815.

3. S. A. Jennings, The structure of the group ring of a p-group over a modular field, Trans. Amer. Math. Soc., 50 (1941) 175–185.

4. N. H. McCoy, Introduction to Modern Algebra, Allyn and Bacon, Boston, rev. ed. 1968.

A COMPACT GRAPH THEOREM

MOON KIM, Seton Hall University

It is known that if the graph $G(f)$ of a function $f: X \to Y$ is compact and X is a Hausdorff space, then f is continuous ([**1**], [**2**]). We obtain here a stronger theorem by using the fact that a compact space may fail to be Hausdorff even though all compact subsets are closed ([**3**]).

In functional analysis, it is well known that boundedness and continuity of an operator are not equivalent if the operator is nonlinear ([**4**]). So the compact graph theorem may be regarded as a nonlinear analog of the closed graph theorem in linear analysis and the following theorem gives a sufficient condition for a compact nonlinear operator to be continuous.

DEFINITION. *$T: X \to Y$ is a proper map if and only if $T^{-1}(C)$ is compact whenever C is.*

THEOREM. *Let X and Y be topological spaces with the property that every compact subset is closed. If the graph $G(f)$ of a function $f: X \to Y$ is compact in $X \times Y$ then f is a proper, continuous, closed map.*

Proof. Let $G(f)$ be compact. Let π_1 and π_2 be the projections of $X \times Y$ on X and on Y respectively. Clearly, X and Range f are compact sets as images of a compact set under π_1 and π_2.

Let π_1^* be the restriction of π_1 to $G(f)$. First, we show that π_1^* is a closed bijection on $G(f) \subset X \times Y$ and hence is a homeomorphism; π_1^* is 1-1; let $\pi_1^*(a,f(a)) = \pi_1^*(b,f(b))$, then $a = b$ and $f(a) = f(b)$, so $(a, f(a)) = (b, f(b))$.

π_1^* is onto; for any $a \in X$, $\pi_1^*(a,f(a)) = a$; π_1^* is closed; let $A \subset G(f)$ be any closed set. Then A is compact, since $G(f)$ is compact. So $\pi_1^*(A)$ is compact and thus closed in X. Thus π_1^* is a homeomorphism of $G(f)$ and X and so $f = \pi_2 \circ \pi_1^{*-1}$ must be continuous.

Next, we show f is a closed map: If A be any closed subset of X, then A is compact and $f(A)$ is compact in Y, so $f(A)$ is closed in Y.

Last, we show f is a proper map: Let C be any compact subset of Range f. Since C is closed, $f^{-1}(C)$ is closed in X. So, $f^{-1}(C)$ is compact whenever C is.

References

1. J. Dugundji, Topology, Allyn and Bacon, Boston, 1966 (Chapter XI).
2. I. Kolodner, The compact graph theorem, Amer. Math. Monthly, 75 (1968) 167.
3. J. Joseph, Continuous functions and spaces in which compact sets are closed, Amer. Math. Monthly, 76 (1969) 1125–1126.
4. M. A. Krasnoselskii, Topological Methods in the Theory of Non-linear Integral Equations, Pergamon Press, New York, 1964.

A NOTE ON DENSITIES OF ORDER STATISTICS

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Let $X_1, X_2, \cdots, X_n$ be independent and identically distributed observations taken from a probability distribution having cumulative distribution function F and density f. Since these random variables are from a continuous distribution, the event that any two or more of these n observations are the same is an event of probability zero. Define Y_k, the kth order statistic, to be the kth largest of $X_1, X_2, \cdots, X_n$. That is, Y_n is the largest observation we obtained, Y_{n-1} is the second largest, $\cdots$, and Y_1 is the smallest observation.

The order statistics $Y_1, Y_2, \cdots, Y_n$ and their distributions are very important in the study of distribution-free or nonparametric statistics as well as in the study of short-cut statistical methods. For example, [2, page 140] the range of the sample, $Y_n - Y_1$, can be used to measure the dispersion of the underlying distribution. Order statistics can be used as well to obtain distribution-free confidence intervals for population quartiles [2, page 13] and other parameters of interest.

It is well known that the density g_k of the kth order statistic Y_k is (at all points x where $F(x)$ is differentiable) given by:

$$(1) \qquad g_k(x) = [F(x)]^{k-1}[1 - F(x)]^{n-k} f(x)\, n!/[(k-1)!\,(n-k)!].$$

From this formula one obtains a heuristic motivation for $g_k(x)$: exactly $k-1$ observations below x, $n-k$ above, and one "right at x." (For this argument using differentials, see [2, page 8].) The formula for g_k can also be obtained using multiple integration [1, page 148].

The density of Y_k, given by (1), will now be derived using mathematical induction. To verify that equation (1) holds when $k = n$, observe that

$$\begin{aligned} G_n(x) &= P[Y_n \leqq x] = P[X_1 \leqq x, \cdots, X_n \leqq x] \\ &= P[X_1 \leqq x] \cdot P[X_2 \leqq x] \cdots P[X_n \leqq x] = [F(x)]^n. \end{aligned}$$

Then

$$g_n(x) = G_n'(x) = n[F(x)]^{n-1} f(x).$$

Thus, (1) follows when $k = n$.

Next, assume that (1) holds for $k = i + 1$ ($i \leqq n - 1$). We show that equation (1) must then hold for $k = i$.

$$\begin{aligned} G_i(x) &= P[Y_i \leqq x] = P[Y_{i+1} \leqq x] + P[Y_i \leqq x < Y_{i+1}] \\ &= G_{i+1}(x) + P[\text{exactly } i \text{ of } X_1, \cdots, X_n \text{ are} \leqq x]. \end{aligned}$$

However P[exactly i of $X_1, X_2, \cdots, X_n$ are $\leqq x$] can be considered as a binomial probability where there are n independent trials and the kth trial is a success if X_k is $\leqq x$ (and a failure otherwise). The probability of a success on the kth trial is, of course, $F(x)$. Thus the above probability is the probability of getting exactly i successes in n binomial trials and is given by the binomial probability

$$[F(x)]^i[1-F(x)]^{n-i}n!/[i!(n-i)!]$$

(see [1, page 86]). Thus

$$G_i(x) = G_{i+1}(x) + [F(x)]^i[1-F(x)]^{n-i}n!/[i!(n-i)!].$$

And

$$\begin{aligned} g_i(x) = G_i'(x) = G_{i+1}'(x) &+ \frac{d}{dx}\{[F(x)]^i[1-F(x)]^{n-i}n!/[i!(n-i)!]\} \\ &= g_{i+1}(x) + i[F(x)]^{i-1}[1-F(x)]^{n-i}f(x)n!/[i!(n-i)!] \\ &\quad - (n-i)[F(x)]^i[1-F(x)]^{n-i-1}f(x)n!/[i!(n-i)!] \\ &= [F(x)]^i[1-F(x)]^{n-i-1}f(x)n!/[i!(n-i-1)!] \\ &\quad + [F(x)]^{i-1}[1-F(x)]^{n-i}f(x)n!/[(i-1)!(n-i)!] \\ &\quad - [F(x)]^i[1-F(x)]^{n-i-1}f(x)n!/[i!(n-i-1)!] \\ &= [F(x)]^{i-1}[1-F(x)]^{n-i}f(x)n!/[(i-1)!(n-i)!]. \end{aligned}$$

Thus, equation (1) holds for $k = i$, and is therefore established for $k = 1, 2, \cdots, n$.

This method can also be used to derive joint densities of order statistics.

References

1. R. V. Hogg and A. T. Craig, Introduction to Mathematical Statistics, Macmillan, New York, 1971.

2. H. A. David, Order Statistics, Wiley, New York, 1970.

SLICING BOXES INTO CUBELETS

SCOTT NIVEN, Washington State University

A known combinatorial problem is as follows: Suppose C is a cube having edges of length 3. By making six planar slices through C, with two parallel slices in each of the three coordinate directions, we can cut up C into 27 cubelets each having edges of length 1. If at any stage in the slicing process it is permissible to restack the pieces already cut, is it possible to cut up C into the 27 cubelets making fewer than six planar slices? The answer to this question is negative, because six different planar slices are necessary to create the six faces of the center cubelet.

In this paper, we generalize the above problem to rectangular parallelotopes in E^n (the general question for rectangular parallelotopes in E^3 was brought to our attention by Robert Jamieson, a Ph. D. student at the University of Washington), and we determine the minimal number of slices needed to cut up such parallelotopes into n-dimensional unit cubes.

$$[F(x)]^i[1 - F(x)]^{n-i} n!/[i!(n - i)!]$$

(see [1, page 86]). Thus

$$G_i(x) = G_{i+1}(x) + [F(x)]^i[1 - F(x)]^{n-i} n!/[i!(n - i)!].$$

And

$$\begin{aligned} g_i(x) = G'_i(x) = G'_{i+1}(x) &+ \frac{d}{dx}\{[F(x)]^i[1 - F(x)]^{n-i} n!/[i!(n - i)!]\} \\ &= g_{i+1}(x) + i[F(x)]^{i-1}[1 - F(x)]^{n-i} f(x) n!/[i!(n - i)!] \\ &\quad - (n - i)[F(x)]^i[1 - F(x)]^{n-i-1} f(x) n!/[i!(n - i)!] \\ &= [F(x)]^i[1 - F(x)]^{n-i-1} f(x) n!/[i!(n - i - 1)!] \\ &\quad + [F(x)]^{i-1}[1 - F(x)]^{n-i} f(x) n!/[(i - 1)!(n - i)!] \\ &\quad - [F(x)]^i[1 - F(x)]^{n-i-1} f(x) n!/[i!(n - i - 1)!] \\ &= [F(x)]^{i-1}[1 - F(x)]^{n-i} f(x) n!/[(i - 1)!(n - i)!]. \end{aligned}$$

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We introduce some notation. If B is a rectangular parallelotope in E^n, we call B a *box*. If a box B has edges of lengths $d_1, d_2, \cdots, d_n$, where the d_i are positive integers, we shall call B a $(d_1, d_2, \cdots, d_n)$-*box*.

Suppose $B_1, B_2, \cdots, B_s$ are boxes in E^n. If a hyperplane P parallel to some coordinate hyperplane is passed through E^n, with the result that each B_i intersected by P is cut into two pieces, we call this process a *slice*. Clearly if some B_i intersected by P has the property that its faces are parallel to the coordinate hyperplanes, then the two pieces into which B_i is cut by P are themselves both boxes.

We now give the generalization of the original problem stated. Let B be a $(d_1, \cdots, d_n)$-box in E^n whose faces are parallel to the coordinate hyperplanes, and suppose we make a sequence of t slices $S_1, S_2, \cdots, S_t$ in the following manner: after any slice S_i resulting in boxes $B_1, B_2, \cdots, B_s$, and before the next slice S_{i+1}, we may translate and rotate any of the B_j to new positions in E^n, subject to the condition that after such translation and rotation the faces of each B_j $(1 \leqq j \leqq s)$ are parallel to the coordinate hyperplanes. It is clear inductively that if B is sliced up in this manner, then the pieces resulting after making any i slices $S_1, S_2, \cdots, S_i$ $(1 \leqq i \leqq t)$ will indeed all be boxes. If the boxes created by this sequence of t slices $S_1, S_2, \cdots, S_t$ are all $(1,1,\cdots,1)$-boxes (i.e., n-dimensional unit cubes), then we call this slicing process a *decomposition of* B, and we say that B *can be decomposed with* t *slices*.
We now state our main result.

THEOREM. *Let B be a $(d_1, d_2, \cdots, d_n)$-box, and let $f(d) = \min\{r \mid 2^r \geqq d,\ r$ an integer$\}$ for any positive integer d. The minimal number of slices needed to decompose B is $f(d_1) + f(d_2) + \cdots + f(d_n)$.*

Before proving this theorem, we state and prove a lemma.

LEMMA. *Suppose $p_1, p_2, \cdots, p_n$ are nonnegative integers. If B is a $(2^{p_1}, 2^{p_2}, \cdots, 2^{p_n})$-box, then B can be decomposed with $p_1 + p_2 + \cdots + p_n$ slices.*

Proof. We induct on $p = p_1 + p_2 + \cdots + p_n$. Thus assume the lemma holds for all $(2^{p_1}, \cdots, 2^{p_n})$-boxes for which $p \leqq k$ $(k \geqq 0)$. We show that the lemma holds for any $(2^{p_1}, \cdots, 2^{p_n})$-box B for which $p = k+1$. Since $k+1 \geqq 1$, there is an i $(1 \leqq i \leqq n)$ such that $p_i \geqq 1$. With one slice, cut B into two boxes B_1 and B_2 which are both $(2^{p_1}, \cdots, 2^{p_i - 1}, \cdots, 2^{p_n})$-boxes. Since $p_1 + \cdots + (p_i - 1) + \cdots + p_n = k$, by inductive hypothesis either B_1 or B_2 can be decomposed with k slices, and thus clearly both B_1 and B_2 can be simultaneously decomposed with k slices. Hence B can be decomposed with $k+1$ slices.

Proof of theorem. We induct on $d = d_1 + d_2 + \cdots + d_n$. Thus assume the theorem holds for all boxes for which $d \leqq k$ (where clearly $k \geqq n$). Let B be any $(d_1, \cdots, d_n)$-box for which $d = k+1$, and suppose t is the minimal number of slices needed to decompose B. In any such decomposition of B with t slices, the first slice results in a $(a_1, a_2, \cdots a_n)$-box B_1 and a $(b_1, b_2, \cdots, b_n)$-box B_2; it is clear that there exists some i $(1 \leqq i \leqq n)$ such that $a_i + b_i = d_i$ and $a_j = b_j = d_j$ for $j \neq i$. Now either $a_i \geqq \frac{d_i}{2}$ or $b_i \geqq \frac{d_i}{2}$, say $a_i \geqq \frac{d_i}{2}$, so that $f(a_i) \geqq f(d_i) - 1$ and thus

$$\sum_{j=1}^{n} f(a_j) \geqq \sum_{j=1}^{n} f(d_j) - 1.$$

Since $a_1 + a_2 + \cdots + a_n \leqq k$, it follows from inductive hypothesis that the minimal number of slices needed to decompose B_1 is $\sum_{j=1}^{n} f(a_j)$, and therefore

$$t \geqq 1 + \sum_{j=1}^{n} f(a_j) \geqq \sum_{j=1}^{n} f(d_j).$$

It only remains to prove that $t \leqq \sum_{j=1}^{n} f(d_j)$. By the lemma, a $(2^{f(d_1)}, \cdots, 2^{f(d_n)})$-box can be decomposed with $f(d_1) + \cdots + f(d_n)$ slices, and since $d_j \leqq 2^{f(d_j)}$ $(1 \leqq j \leqq n)$, it is obvious that B can also be decomposed with this many slices.

BOOK REVIEWS

EDITED BY ADA PELUSO AND WILLIAM WOOTON

Materials intended for review should be sent to: Professor Ada Peluso, Department of Mathematics, Hunter College of CUNY, 695 Park Avenue, New York, New York 10021, or to Professor William Wooton, 1495 La Linda Drive, Lake San Marcos, California 92069. A boldface capital **C** *in the margin indicates that a review is based in part on classroom use.*

A Catalog of Special Plane Curves. By J. Dennis Lawrence. Dover, New York, 1972. xi + 218 pp. $3.00.

While it may no longer be true, the college mathematical experience of many mathematicians in the "over thirty" category was liberally sprinkled with "beautiful curves." Texts which confined ε-δ arguments to only two pages in the entire book did not fail to include material on "higher" analytic or differential geometry which has disappeared from "modern" texts. Most such texts also included a brief catalog of higher plane curves in the endpapers, each curve labeled with a Latinized name or that of a great mathematician. I can still see one teacher of freshman calculus holding his watch chain while explaining a catenoid, and the first college math teacher I was exposed to had only one joke, which he told at least four times during the semester: "Do you know what shape a kiss is?" — "E-lip-tickle!"

Textbooks and mathematical fads have changed, but the special plane curves still are intriguing to many students and teachers of mathematics — even if they now are considered "old math." This appropriately titled book will be of interest and value to such an audience. It is a compendium of formulae and analytic information for over 60 special plane curves, including computer generated plots, parametric and polar equations, and relationships to the other curves.

Of the seven chapters, the last five actually form the catalog. Each successive chapter covers in varying detail curves of higher complexity. Aside from the completeness of the formulae which are included, the plots are the unique feature of this book — unfortunately marred by "too large" a step size (locally wiggly curves) and lack

$$\sum_{j=1}^{n} f(a_j) \geqq \sum_{j=1}^{n} f(d_j) - 1.$$

Since $a_1 + a_2 + \cdots + a_n \leqq k$, it follows from inductive hypothesis that the minimal number of slices needed to decompose B_1 is $\sum_{j=1}^{n} f(a_j)$, and therefore

$$t \geqq 1 + \sum_{j=1}^{n} f(a_j) \geqq \sum_{j=1}^{n} f(d_j).$$

It only remains to prove that $t \leqq \sum_{j=1}^{n} f(d_j)$. By the lemma, a $(2^{f(d_1)}, \cdots, 2^{f(d_n)})$-box can be decomposed with $f(d_1) + \cdots + f(d_n)$ slices, and since $d_j \leqq 2^{f(d_j)}$ $(1 \leqq j \leqq n)$, it is obvious that B can also be decomposed with this many slices.

BOOK REVIEWS

EDITED BY ADA PELUSO AND WILLIAM WOOTON

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Of the seven chapters, the last five actually form the catalog. Each successive chapter covers in varying detail curves of higher complexity. Aside from the completeness of the formulae which are included, the plots are the unique feature of this book — unfortunately marred by "too large" a step size (locally wiggly curves) and lack of any scale reference. With our programmable calculator and X-Y plotter, we were able to duplicate many of the curves "more smoothly" and to detect some misleading features of the plots provided. Due to a change of scale for one axis, the witches on p. 93 are at odds with the diagram on p. 91. The plot labeled "j" on p. 38 is incorrect, and a number of similar minor errors appear.

Chapter 1 contains a discussion of various background materials including some coordinate systems which may not be familiar, some analytic geometry, and elementary differential geometry. The notation used here and throughout the book shows no trace of recent changes of approach in calculus texts. Several definitions, including those of "curve" and "arc," are not consistent with modern approaches, but the book is readable and useable by anyone who has "mastered analytic geometry and calculus." Chapter 2 describes ten types of curves which may be derived from a base curve — evolute, involute, pedal, etc. This material is used in the subsequent chapters and tables. A lengthy bibliography of accessible and interesting references is included. It would appear that the author drew heavily from the article *Curves, Special* in the Encyclopedia Brittanica, written by R. C. Archibald and Nathan A. Court, but that tersely written source is greatly expanded upon in this book.

While names and dates are frequently cited, there is no narrative which would answer questions such as: Who was Tchirnhausen? Why is the trisectrix of Maclaurin called a trisectrix? Who was Freeth, and why does he have a nephroid named after him? Why is a nephroid called a nephroid? In other words, the book is not a treatise, but it is an interesting, potentially useful resource which will help to keep an interesting area of "old math" alive.

GEORGE C. DORNER, William Rainey Harper College

The Fascination of Groups. By F. J. Budden. Cambridge University Press, London, England, 1972. xv + 545 pp. $18.50.

This book is a leisurely paced introduction to the theory of groups, intended primarily as a reference for secondary school teachers in preparing a high school course in group theory or for undergraduate students who want more insight into the internal structure of finite groups. Furthermore, as is stated by the author, this book could be used by any student who is comfortable with the topics covered in high school courses in algebra and geometry plus possibly an understanding of congruences. The wide accessibility of this book is achieved by centering the development of group theory upon the example rather than the proof. The author, writing from his own teaching experience with precollege students, has used examples to develop and motivate the definitions, the statements of theorems, as well as the proofs. Moreover, he has done this in such an inexhaustible fashion that each succeeding stage of the development becomes a necessity rather than a convenience. For example, in the section on isomorphism he gives thirteen different contexts in which the Klein 4-Group can arise. After reading such a list, it is inconceivable for us to see how anyone could question the wisdom of having the concept of isomorphism. In fact, in this context, the concept seems to be a light of wisdom in the midst of any scale reference. With our programmable calculator and X-Y plotter, we were able to duplicate many of the curves "more smoothly" and to detect some misleading features of the plots provided. Due to a change of scale for one axis, the witches on p. 93 are at odds with the diagram on p. 91. The plot labeled "j" on p. 38 is incorrect, and a number of similar minor errors appear.

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The material presented in this book can be easily divided into four major parts — the foundations, the basic properties of groups, the basic theory of subgroups and group homomorphisms, and finally some applications. In the first seven chapters the author presents the background material necessary for the later development. Here he discusses functions and algebraic structures, binary operations, closure, commutativity and associativity, identities, and inverses, both in the terms of precise mathematical terminology and in the more informal terms of a discussion of the import of each idea. Next, in chapters 8 to 13, the student is given the basic properties of groups along with a detailed presentation of the basic examples of finite groups — the symmetric groups, cyclic groups, and dihedral groups. Following this, Mr. Budden develops the more complex ideas of elementary group theory including Lagrange's Theorem, generators, direct products, cosets, homomorphisms, normal subgroups, and quotient groups. Finally, in Chapters 23 to 26, the author presents four situations (three of which are nonmathematical) in which groups naturally arise. The first chapter concerns groups and music and emphasizes the relationship of the cyclic group of twelve elements to the equal tempered scale. This is followed by a discussion of the relationship of campanology and the symmetric groups. Both of these topics are interesting and to our knowledge do not appear in other recent works in group theory. Finally, the book is completed with a chapter on groups and geometry (including Pappus' Theorem and Desargues' Perspective Triangle Theorem) and a chapter which discusses the relationship between designers' patterns and the cyclic and dihedral groups.

It is our opinion that a word must be included concerning the somewhat unusual title of this book. Although the author is careful to include examples of groups which arise outside of group theory as well as an appendix on the applications of group theory, it is our opinion that the fascination of group theory for Mr. Budden is primarily the internal structure of the group. In other words, from reading this book we are certain that the study of the inner-relationships of the elements of a group is a source of fascination and enjoyment for Mr. Budden, and it is exactly these attitudes that he has related in his treatment.

In conclusion we mention that as much as there is a conspicuous use of examples in this book, there is also nearly no mention of the theory of groups acting on sets. Except in his development of Cayley's Theorem, which Mr. Budden does not prove in generality, any precise mention of these considerations is avoided. Moreover, there is also no indication made by the author as to the advisability of introducing this attitude of development into a presentation of group theory to young students, an attitude of development which, in our opinion, unifies much of the material presented here.

J. A. LOUSTAU, Hunter College of CUNY

PROBLEMS AND SOLUTIONS

EDITED BY ROBERT E. HORTON, Los Angeles Valley College

ASSOCIATE EDITOR, J. S. FRAME, Michigan State University

Readers of this department are invited to submit for solution problems believed to be new that may arise in study, in research, or in extra-academic situations. Problems may be submitted from any branch of mathematics and ranging in subject content from that accessible to the talented high school student to problems challenging to the professional mathematician. Proposals should be accompanied by solutions, when available, and by any information that will assist the editor. Ordinarily, problems in well-known textbooks should not be submitted.

The asterisk (*) will be placed by the problem number to indicate that the proposer did not supply a solution. Readers' solutions are solicited for all problems proposed. Proposers' solutions may not be "best possible" and solutions by others will be given preference.

Solutions should be submitted on separate, signed sheets. Figures should be drawn in India ink and exactly the size desired for reproduction.

Send all communications for this department to Robert E. Horton, Los Angeles Valley College, 5800 Fulton Avenue, Van Nuys, California 91401.

To be considered for publication, solutions should be mailed before August 1, 1974.

PROPOSALS

894. *Proposed by J. A. H. Hunter, Toronto, Canada.*

In this alphametic we naturally have a prime $MATHS$!

$$\begin{array}{rrrrr} & T & H & I & S \\ & & M & A & N \\ & T & H & I & S \\ & & & I & S \\ \hline M & A & T & H & S \end{array}$$

895. *Proposed by C. F. Pinzka, University of Cincinnati.*

Evaluate

$$I = \int \sqrt{\sec^2 x + a}\, dx, \quad a \geqq 0.$$

896. *Proposed by Stephen B. Maurer, Phillips Exeter Academy, New Hampshire.*

A Pythagorean triplet is a triple (a, b, c) of integers such that $a^2 + b^2 = c^2$. Prove that there are infinitely many Pythagorean triplets of the form $(a, a+1, c)$.

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897. *Proposed by C. S. Venkataraman, Sree Kerala Varma College, Trichur, South India.*

If $a_1, a_2, a_3, \cdots, a_k$ are the numbers prime to and not greater than n, prove that

$$\sum_{i=1}^{k} \left(\frac{a_i}{n - a_i}\right) \geqq \phi(n).$$

898. *Proposed by Roger D. H. Jones, University of Georgia.*

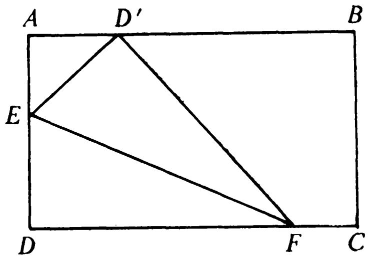
There are five points associated with every triangle: the orthocenter, the centroid, the incenter, the circumcenter, and the nine-point center. Prove that if any two of these coincide the triangle is equilateral.

899. *Proposed by Charles W. Trigg, San Diego, California.*

The arithmetic mean of the twin primes 5 and 7 is the triangular number 6. Are there any other twin primes with a triangular mean?

900. *Proposed by Murray S. Klamkin, Ford Motor Company, and Seymour Papert, Massachusetts Institute of Technology.*

A long sheet of rectangular paper $ABCD$ is folded such that D falls on AB producing a smooth crease EF with E on AD and F on CD (when unfolded). Determine the minimal area of triangle EFD by elementary methods.



Errata. Proposal 878 should read: $1 + x > \left(1 + \frac{x}{2^k + 1}\right)^k$ when $0 < x < 2^k + 1$.

QUICKIES

From time to time this department will publish problems which may be solved by laborious methods, but which with the proper insight may be disposed of with dispatch. Readers are urged to submit their favorite problems of this type, together with the elegant solutions and the source, if known.

Q588. Show that the sequence $\{\tan (n)\}_{n \geqq 1}$ is divergent.

[*Submitted by Bernard C. Anderson*]

Q589. If a man takes h hours to make a certain trip, how much faster must he travel to make a trip m miles longer in the same time?

[*Submitted by C. F. Pinzka*]

Q590. $O-ABCDE$ is a regular pentagonal pyramid such that $\angle AOB = 60°$. Find $\angle AOC$.

[*Submitted by Murray S. Klamkin*]

Q591. The number of distinct primes that divide the positive integer n is exactly equal to $\log_2 (\sum_{d|n} \mu^2(d))$ where μ is the Möbius Function.

[*Submitted by L. C. Eggan*]

Q592. If a, m and n are positive integers and n is odd, prove that the greatest common divisor of $a^n - 1$ and $a^m + 1$ is not greater than 2.

[*Submitted by Erwin Just*]

(Answers on page 114)

SOLUTIONS

Late Solutions

Mary F. Turner, Mathematics and Science Center, Glen Allen, Viriginia: **859**; *H. Marlon Hewit, Reedley, California:* **859.**

A Congruence Cryptarithm

866. [May, 1973] *Proposed by Richard L. Breisch, Pennsylvania State University.*

Solve the congruence cryptarithm $LIFE \equiv SIZE \pmod{ELS}$ in base 6 with E, L and S nonzero.

Solution by Kenneth M. Wilke, Topeka, Kansas.

Solution: All computations are made in base six. The congruence implies $|LIFE - SIZE| = k \cdot ELS$ for some integer k. Hence we have either

$$\begin{array}{r} LIFE \\ -\ SIZE \\ \hline ABCO \end{array} \quad \text{or} \quad \begin{array}{r} SIZE \\ -\ LIFE \\ \hline ABCO \end{array} \quad \text{where } ABCO > 0$$

depending upon whether $S < L$ or $S > L$. $C \neq 0$ since F and Z are distinct. Furthermore the subtraction of I from I implies that either $B = 0$ and $A = |S - L|$ or $B = 5$ and $A + 1 = |S - L|$. Finally if $S = 1$ or 5, k is a multiple of 10; if $S = 2$ or 4, k is a multiple of 3; and if $S = 3$, k is even. Using these criteria the possible choices of ELS and $ABCO$ are limited to:

E L S	A B C O
1 2 5	2 5 4 0
1 5 3	1 5 3 0
1 5 4	0 5 5 0
2 5 3	1 5 4 0
3 4 2	1 5 1 0
3 5 1	3 5 1 0

Of these only $E\,L\,S = 1\;5\;4$ and $E\,L\,S = 3\;4\;2$ yield solutions which are $5021 \equiv 4031 \pmod{154}$ and $4103 \equiv 2153 \pmod{342}$ respectively.

Also solved by Richard Anders, Great Neck, N. Y.; M. G. Greening, University of New South Wales, Australia; J. A. H. Hunter, Toronto, Canada; Vaclav Konecny, San Jose, California; Charles Linett, Fruitland, Maryland; Robert F. Sutherland, Bridgewater State College, Massachusetts; Charles W. Trigg, San Diego, California; Mary F. Turner, Glen Allen, Virginia; Kenneth M. Wilke, Topeka, Kansas; R. F. Wardrop, Central Michigan University; William S. Wood and Genee Logue (jointly), Pennsylvania State University; and the proposer.

Property of an Interior Point

867. [May, 1973] *Proposed by L. Carlitz, Duke University.*

Let P be a point in the interior of the triangle ABC. Let R_1, R_2, R_3, denote the distances of P from the vertices of ABC and let r_1, r_2, r_3, denote the distances from P to the sides of ABC. Show that

$$\Sigma\, r_1R_2R_3 \geqq 12r_1r_2r_3, \tag{1}$$

$$\Sigma\, r_1R_1^2 \geqq 12r_1r_2r_3, \tag{2}$$

$$\Sigma\, r_2^2r_3^2R_2R_3 \geqq 12r_1^2r_2^2r_3^2. \tag{3}$$

In each case there is equality if and only if ABC is equilateral and P is the center of ABC.

I. *Solution by M. G. Greening, University of New South Wales, Australia.*

(A) $R_1R_2R_3 \geqq 8r_1r_2r_3$, equality holding if and only if ABC is equilateral and P is the center of ABC.

(1) $\Sigma\, r_1R_2R_3 \geqq 3(r_1r_2r_3)^{1/3}(R_1R_2R_3)^{2/3}$ by the *AM/GM inequality*

$$\geqq 12\,r_1r_2r_3 \text{ from (A).}$$

(2) $$\Sigma\, r_1R_1^2 \geqq 3(r_1r_2r_3)^{1/3}(R_1R_2R_3)^{2/3}$$

$$\geqq 12\,r_1r_2r_3 \text{ as in (1).}$$

(3) $$\Sigma\, r_2^2r_3^2R_2R_3 \geqq 3(r_1r_2r_3)^{4/3}(R_1R_2R_3)^{2/3}$$

$$\geqq 12(r_1r_2r_3)^2 \text{ using (A).}$$

In each case equality clearly holds if the conditions are as in (A) for then $r_iR_jR_k = r_jR_kR_i$, etc. On the other hand, as these conditions are sufficient for equality in the first inequality of each of (1), (2) and (3) and necessary for equality in the second, they are necessary for each of the three stated inequalities; i.e., there is equality in each case if and only if ABC is equilateral and P is the center.

II. *Solution by Murray S. Klamkin, Ford Motor Company.*

The three inequalities are special cases of

$$\sum_{\text{cyclic}} \frac{R_1^i R_2^j R_3^k}{r_1^u r_2^v r_3^w} \geqq 3\left\{\frac{R_1R_2R_3}{r_1r_2r_3}\right\}^{m/3} \geqq 3\cdot 2^m$$

where $i+j+k=u+v+w=m\geqq 0$. The left hand inequality follows immediately from the A.M.-G.M. inequality while the right hand inequality follows from the known inequality $R_1R_2R_3 \geqq 8r_1r_2r_3$ with equality *iff* ABC is equilateral and P is the center [see O. Bottema, et al., *Geometric Inequalities*, Walters-Noordhoff, Groningen, 1969, p. 111].

REMARK: We can obtain a stronger identity by using (loc. cit).

$$R_1R_2R_3 \geqq r_1r_2r_3/\Pi \sin A/2.$$

Also, by using $(x+y+z)/3 \geqq \{\sum yx/3\}^{1/2}$, we can augment the proposed inequalities to

1. $$\left\{\sum \frac{R_1}{r_1}\right\}^2 \geqq 3\sum\frac{R_2R_3}{r_2r_3} \geqq 36,$$

2. $$\left\{\sum \frac{1}{r_1R_1^2}\right\}^2 \geqq 3\sum\frac{1}{r_2r_3R_2^2R_3^2} \geqq 36,$$

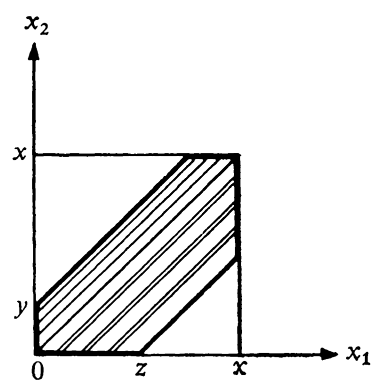
3. $$\left\{\sum r_1^2R_1\right\}^2 \geqq 3\sum r_2^2r_3^2R_2R_3 \geqq 36\, r_1^2r_2^2r_3^2.$$

Also solved by Graham Lord, Temple University; Phil Tracy, Liverpool, New York; and the proposer.

A Problem of Meeting

868.* [May, 1973] *Proposed by J. A. H. Hunter, Toronto, Canada.*

Two men undertake to arrive at a rendezvous independently, but each would arrive sometime between noon and x minutes after noon. One promises to wait y minutes, the other z minutes, but neither will stay beyond x minutes after noon. What is the chance that they will meet?



Solution by Vaclav Konecny, San Jose, California.

Let the men arrive at the moments x_1 and x_2, respectively; they will meet if $0 < x_2 - x_1 < y$ or $0 < x_1 - x_2 < z$. Using the geometrical representation we get the probability p that they will meet, from the figure on page 110,

$$p = \frac{x^2 - (x-y)^2/2 - (x-z)^2/2}{x^2}.$$

Also solved by Carl A. Argila, De La Salle College, Philippines; D. W. Brown, Princeton, New Jersey; Barry Edenbaum, Brooklyn, New York; Abraham L. Epstein, Bedford, Massachusetts; G. Farr, Capricornia Institute of Advanced Education, Rockhampton, Australia; W. W. Funkenbusch, Michigan Technological University; Michael Goldberg, Washington, D. C.; Gene Hartman, New York, New York; Warren Frank Lamboy, Carroll College, Wisconsin; Ross Renner, University of the South Pacific, Fiji; Rina Rubenfeld, New York City Community College; Jan B. Schipmolder, Sunnyvale, California; Joseph H. Silverman, White Plains, New York; Thomas Spencer, Trenton State College, New Jersey; R. S. Stacy, Albuquerque, New Mexico; Summer Class 1973 In Mathematics 483C, Probability and Statistics, Southern Illinois University; Harold Ziehms, Monterey, California.

A Coefficient of an Expansion

869. [May, 1973] *Proposed by Roy Dubisch, East African Regional Mathematics Program, Addis Ababa, Ethiopia.*

Prove that in the expansion of $(x-1)(x-2)\cdots(x-k)$ the coefficient of x^{k-2} is $k(k+1)(k-1)(3k+2)/24$.

I. *Solution by F. D. Parker, St. Lawrence University.*

Solution: Clearly the required coefficient is given by $C = \prod_{ij}$ where i and j range from 1 to k, but $i \neq j$.

Consider the sum

$$S = 1(1+2+\cdots k) + 2(1+2+\cdots k) + \cdots + k(1+2+\cdots k)$$
$$= (1+2+\cdots k)(1+2+\cdots k) = \frac{k^2(k+1)^2}{4}.$$

If we subtract $1^2 + 2^2 + \cdots k^2 = \dfrac{k(k+1)(2k+1)}{6}$ from S, the result will be $2C$. Consequently

$$2C = \frac{k^2(k+1)^2}{4} - \frac{k(k+1)(2k+1)}{6} = \frac{k(k+1)(k-1)(3k+2)}{12}, \text{ and}$$

$$C = \frac{k(k+1)(k-1)(3k+2)}{24}.$$

II. *Solution by Lawrence A. Ringenberg, Eastern Illinois University.*

The required coefficient C_k is the sum of the elements above the main diagonal of the following symmetric matrix:

$$\begin{bmatrix} 1\cdot 1 & 1\cdot 2 & 1\cdot 3 & \cdots & 1\cdot k \\ 2\cdot 1 & 2\cdot 2 & 2\cdot 3 & \cdots & 2\cdot k \\ \cdots & & & & \\ k\cdot 1 & k\cdot 2 & k\cdot 3 & \cdots & k\cdot k \end{bmatrix}.$$

Then $2C_k = \left[\frac{k(k+1)}{2}\right]^2 - \frac{k(k+1)(2k+1)}{6}$ and $C_k = \frac{k(k+1)(k-1)(3k+2)}{24}$.

Also solved by Richard Anders, Great Neck, New York; Carl A. Agila, De La Salle College, Manila, Philippines; Gladwin Bartel, La Junta, Colorado; Melvin Billik, Midland High School, Michigan; M. T. Bird, California State University, San Jose; Richard L. Breisch, Alamogordo, New Mexico; Brother Alfred Brousseau, St. Mary's College, California; L. Carlitz, Duke University; Jacques Chone, Lycée d'Etat Mixte, Thiers, France; Robert H. Cornell, Exeter, New Hampshire; Stephen C. Currier, Jr., Pennsylvania State University, Altoona; Charles A. DeCarlucci, Pennsylvania State University; Santo M. Diano, Havertown, Pennsylvania; Ragnar Dybvik, Tingvoll, Norway; Hugh M. Edgar, California State University, San Jose; Walter O. Egerland, Aberdeen Proving Ground, Maryland; Abraham L. Epstein, Bedford, Massachusetts; Stanley Fox, City College of New York; Ralph Garfield, The College of Insurance, New York City; Richard A. Gibbs, Fort Lewis College, Colorado, Michael Goldberg, Washington, D. C.; M. G. Greening, University of New South Wales, Australia; Richard A. Groeneveld, Iowa State University; Eino Halminen, Helsinki, Finland; J. R. Hanna, University of Wyoming; Gene Hartman, New York City; Howard Hiller, Cornell University; John Homer, South Charleston, West Virginia; John M. Howell, Littlerock, California; Ralph Jones, University of Massachusetts; Vaclav Konecny, San Jose, California; Henry S. Lieberman, Boston, Massachusetts; Charles Linett, Fruitland, Maryland; Graham Lord, Temple University; Robert Meyer, University of Nebraska; C. F. Pinzka, University of Cincinnati; Willis B. Porter, New Iberia, Louisiana; Bob Prielipp, University of Wisconsin, Oshkosh; Lois J. Reed, Longwood College, Farmville, Virginia; M. Rodeen, San Mateo, California; Rina Rubenfeld, New York City Community College; Jan B. Schipmolder, Sunnyvale, California; Joseph Silverman, White Plains, New York; R. S. Stacy, Manzano High School, Albuquerque, New Mexico; Eric Sturley, Southern Illinois University, Edwardsville; James G. Troutman, York College of Pennsylvania; Wolf R. Umbach, Rottdorf, Germany; Edward T. H. Wang, University of Waterloo, Ontario, Canada; Alan Wayne, Holiday, Florida; Betty Ann Whitted, Bennett College, Greensboro, North Carolina; Kenneth M. Wilke, Topeka, Kansas; Atila Yanik, Illinois Institute of Technology, Chicago; Tao-Cheng Yit, Carleton University, Ottawa, Canada; K. L. Yocom, South Dakota State University; and the proposer.

Perfect Numbers

870. [May, 1973] *Proposed by Everett Casteel, Bethel College, Minnesota.*

Given the number $2^{n-1}(2^n-1)$ where n is an odd integer greater than or equal to 3, prove that $2^{n-1}(2^n-1) \equiv 1 \pmod 9$.

Solution by Lois J. Reid, Langwood College, Virginia.

Since n is odd, assume $n = 2k+1$, where $k \in Z$, $k \geqq 1$, and prove $2^{2k}(2^{2k+1}-1) \equiv 1 \pmod 9$. But $k \in Z$ can be written in the form (i) $k = 3j$, (ii) $k = 3j+1$, or (iii) $k = 3j+2$, where $j \in Z$, $j \geqq 0$. So we consider each case, recalling that $2^6 \equiv 1 \pmod 9$ and find that

(i) $2^{6j}(2^{6j+1}-1) \equiv 1(2-1) \equiv 1 \pmod 9$,
(ii) $2^{6j+2}(2^{6j+3}-1) \equiv 2^2(2^3-1) \equiv 4.7 \equiv 1 \pmod 9$,
(iii) $2^{6j+4}(2^{6j+5}-1) \equiv 2^4(2^5-1) \equiv 7(5-1) \equiv 1 \pmod 9$.

Also solved by Richard Anders, Great Neck, New York; Gladwin Bartel, La Junta, Colorado; Richard A. Bauer, Port Orchard, Wisconsin; E. D. Bender, Bradley University, Peoria, Illinois; G. E. Bergum, South Dakota State University; M. T. Bird, California State University, San Jose; Richard L. Breisch, Alamogordo, New Mexico; David C. Brooks, Rob Emerson, Janiece McDonald, Gail Wickre and Rob Emerson (jointly) Seattle Pacific College; Jacques Chone, Lycée d'Etat Mixte, Thiers, France; Stephen C. Currier, Jr., Pennsylvania State University, Altoona; Mohammed Dadashzadeh, Massachusetts Institute of Technology; E. F. Ecklund, Jr., Northern Illinois University; Hugh M. Edgar, California State University, San Jose; W. O. Egerland, Aberdeen Proving Ground, Maryland; Marjorie Fitting, California State University, San Jose; Stanley Fox, City College of New York; William F. Fox, Moberly Area Junior College; Ralph Garfield, The College of Insurance, New York City; Richard A. Gibbs, Fort Lewis College, Colorado; Michael Goldberg, Washington, D. C.; M. G. Greening, University of New South Wales, Australia; Olive Hobbs, Kent, Ohio; Ralph Jones, University of Massachusetts; Vaclav Konecny, San Jose, California; Lew Kowarski, Morgan State College, Baltimore, Maryland; Henry S. Lieberman, Boston, Massachusetts; Peter W. Lindstrom, St. Anselm's College, New Hampshire; Charles Linett, Fruitland, Maryland; Graham Lord, Temple University; Robert Meyer, University of Nebraska; Thomas E. Moore, Bridgewater State College, Massachusetts; Roger Osborn, University of Texas at Austin; F. J. Papp, University of Lethbridge, Alberta, Canada; F. D. Parker, St. Lawrence University; C. F. Pinzka, University of Cincinnati; Willis B. Porter, New Iberia, Louisiana; Bob Prielipp, University of Wisconsin at Oshkosh; Gerson B. Robison, State University College, New Paltz, New York; M. Rodeen, San Mateo, California; Rina Rubenfeld, New York City Community College; Erwin Schmid, Washington, D. C.; Joseph Silverman, White Plains, New York; Robert S. Stacy, Manzano High School, Albuquerque, New Mexico; Eric Sturley, Southern Illinois University; Charles W. Trigg, San Diego, California; Wolf R. Umbach, Rottdorf, Germany; R. F. Wardrop, Central Michigan University; Edward T. H. Wang, University of Waterloo, Ontario, Canada; Alan Wayne, Holiday, Florida; Kenneth M. Wilke, Topeka, Kansas; H. H. Wong, Ohio State University; Dale Woods and William T. Wood (jointly), Northeast Missouri State University; Tao-Cheng Yit, Carleton University, Ottawa, Canada; K. L. Yocom, South Dakota State University; and the proposer.

Property of an Abelian Group

871. [May, 1973] *Proposed by Donald P. Minassian, Butler University, Indiana.*

A subgroup H of a fully ordered group G is convex if H contains all g in G such that $h \leqq g \leqq k$ whenever $h \leqq k$ are both in G. Let H be a proper subgroup of an abelian group G whose only element of finite order is the identity. Show that G admits an ordering under which H is not convex.

Solution by Phillip Schultz, The University of Western Australia.

If H is a pure subgroup of G, (i.e., $nG \cap H = nH$ for all integers n), then G has a rationally independent set $X \cup Y$, $Y \neq \varnothing$, such that X is a rationally independent set in H. Let n be the least ordinal such that $|X \cup Y| < |n| 2^{\aleph_0}$; let J be the group direct sum $\bigoplus_{i<n} R_i$, where R_i is isomorphic to the additive group of real numbers, with the usual ordering; and order J lexicographically. Now let $x \in X$, $y \in Y$ and let K be a Hamel basis for J containing the elements 1 and π of R_1. Then there is a 1-1 correspondence between $X \cup Y$ and K which maps $x \mapsto 1$, $y \mapsto \pi$ and can be extended to a group embedding of G into J. With the ordering inherited from J, G is a fully ordered group such that $x < y < 4x$, with $x, 4x \in H$, $y \notin H$.

If H is not pure, let $a \in G$ such that $a \notin H$ but $na \in H$. Then in any order on G which makes G a fully ordered group (for example, G could be embedded in a direct sum of copies of the real numbers as described in the first paragraph), $na < (n+1)a < 2na$, where na, $2na \in H$, $(n+1)a \notin H$.

Also solved by the proposer.

A Summation Problem

872. [May, 1973] *Proposed by Warren Page, New York City Community College.*

Let $x = \sum_{i=1}^{n} \alpha_i u_i^2$, where u_i are integers and $\alpha_1 = 1$. Prove that for every natural number k one has $x^{2^k} = \sum_{i=1}^{n} \alpha_i w_i^2$, where w_i are integers.

Solution by Bob Prielipp, University of Wisconsin, Oshkosh.

It suffices to prove that if $y = \sum_{i=1}^{n} \gamma_i r_i^2$, where r_i are integers and $\gamma_1 = 1$ then $y^2 = \sum_{i=1}^{n} \alpha_i s_i^2$, where s_i are integers. (The solution is completed by induction on k, where k is given in the statement of the problem.) Let $y = \sum_{i=1}^{n} \gamma_i r_i^2$. Then

$$y^2 = (r_1^2 + \gamma_2 r_2^2 + \gamma_3 r_3^2 + \cdots + \gamma_n r_n^2)^2 = (r_1^2 - \gamma_2 r_2^2 - \gamma_3 r_3^2 - \cdots - \gamma_n r_n^2)^2$$
$$+ \gamma_2(2r_1r_2)^2 + \gamma_3(2r_1r_3)^2 + \cdots + \gamma_n(2r_1r_n^2)^2.$$

Also solved by M. T. Bird, California State University at San Jose; John Homer, South Charleston, West Virginia; Vaclav Konecny, San Jose, California; Rina Rubenfeld, New York City Community College; Phil Tracy, Liverpool, New York; and the proposer.

ANSWERS

A588. Assume $\lim_{n\to\infty} \tan(n) = p$ for some p in R. Then by letting $n \to \infty$ in the identity, $[1 - \tan(n)\tan 1]\tan(n-1) = \tan(n) - \tan(1)$, one obtains $[1 - p\tan(1)]p = p + \tan(1)$. Consequently $p^2 = -1$, a contradiction!

A589. Suppose he starts m miles behind a person making the regular trip at the regular speed. Considering relative velocities, he must travel m/h miles per hour faster to catch up in h hours.

A590. By symmetry $\angle AOC = \angle ABC = 108°$.

A591. If t denotes the number of distinct primes in n, then the problem merely asserts that the function $G(n) = \sum_{d|n} \mu^2(d)$ has the value 2^t. Now μ and hence μ^2 is multiplicative, so it follows that G is also. It suffices then to evaluate G at prime powers: $G(p^\alpha) = \mu^2(1) + \mu^2(p) + \cdots + \mu^2(p^\alpha) = 2$. Thus

$$G(p_1^{\alpha_1} \cdots p_t^{\alpha_t}) = \prod_{j=1}^{t} G(p_j^{\alpha_j}) = 2^t.$$

A592. Let $d = [a^n - 1, a^m + 1]$. Then $a^n = kd + 1$ and $a^m = ld + 1$. Therefore $a^{mn} = (a^n)^m = td + 1$ and $a^{mn} = (a^m)^n = ud - 1$. Thus $td + 1 = ud - 1$ or $(u - t)d = 2$. It follows that $d = 1$ or $d = 2$.

(Quickies on pages 107–108)

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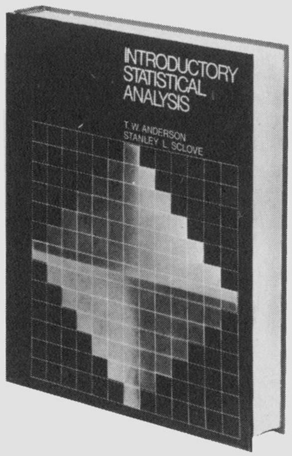

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